POLYNOMIALS WITH GENERAL $\mathbf{C}^2$–FIBERS ARE VARIABLES. I.

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ABSTRACT. Suppose that X' is a smooth affine algebraic variety of dimension 3 with $H_3(X') = 0$ which is a UFD and whose invertible functions are constants. Suppose that Z is a Zariski open subset of X which has a morphism $p : Z \to U$ into a curve U such that all fibers of p are isomorphic to $\mathbf{C}^2$. We prove that X' is isomorphic to $\mathbf{C}^3$ iff none of irreducible components of $X' \backslash Z$ has non-isolated singularities. Furthermore, if X' is $\mathbf{C}^3$ then p extends to a polynomial on $\mathbf{C}^3$ which is linear in a suitable coordinate system. As a consequence we obtain the fact formulated in the title of the paper.

1. INTRODUCTION

We say that a nonconstant polynomial on $\mathbf{C}^n$ is a variable if it is linear in a suitable polynomial coordinate system on $\mathbf{C}^n$. Classification of such polynomials is a difficult and important problem which is solved only for $n = 2$. In 1961 Gutwirth [Gu] proved the following fact which was later reproved by Nagata [Na]: every polynomial p in two complex variables whose general fibers are isomorphic to $\mathbf{C}$ (that is, there exists a finite subset S of $\mathbf{C}$ such that for every $c \in \mathbf{C} \setminus S$ the fiber $p^{-1}(c)$ is isomorphic to $\mathbf{C}$) is a variable. In 1974–1975 Abhyankar, Moh, and Suzuki showed that a much stronger fact holds: every irreducible polynomial in two complex variables, whose zero fiber is isomorphic to $\mathbf{C}$, is variable [AbMo], [Su]. The Embedding conjecture formulated by Abhyankar and Sathaye [Sa1] suggests that the similar fact holds in higher dimensions:

Every irreducible polynomial p in n complex variables with a fiber isomorphic to $\mathbf{C}^{n-1}$ is a variable.

It seems that in the full generality the positive answer to the Embedding conjecture is not feasible in the near future but there is some progress for $n = 3$. In this dimension A. Sathaye, D. Wright, and P. Russell proved some special cases of this conjecture ([Sa1], [Wr], [RuSa], see also [KaZa1]). Then M. Koras and P. Russell

proved the Linearization conjecture for $n = 3$ [KoRu2], [KaKoM-LRu] which implies the following theorem:

if p is an irreducible polynomial on $\mathbf{C}^3$ such that it is quasi-invariant with respect to a regular $\mathbf{C}^*$-action on $\mathbf{C}^3$ and its zero fiber is isomorphic to $\mathbf{C}^2$, then p is a variable. [1]

This paper and the next joint paper of the author with M. Zaidenberg [KaZa2] contain another step in the direction of the Embedding conjecture – we prove the analogue of the Gutwirth theorem in dimension 3, i.e. every polynomial with general $\mathbf{C}^2$–fibers is a variable. It is worth mentioning that a special case of this theorem follows from more general results of Miyanishi [Miy1] and Sathaye [Sa2] (we are grateful to P. Russell who drew our attention to the paper of Sathaye). They showed that if each fiber $p^{-1}(c)$, $c \in \mathbf{C}$ of a polynomial $p \in \mathbf{C}[x, y, z]$ is isomorphic to $\mathbf{C}^2$ and its generic fiber is also plane [2] then p is a variable.

In fact, in our papers the analogue of the Gutwirth theorem in dimension 3 is also a consequence of the following more general result.

Main Theorem. *Let X' be an affine algebraic variety of dimension 3 such that X' is a UFD [3], all invertible functions on X' are constants, and*

(1) the Euler characteristic of X' is $e(X') = 1$;

(2) there exists a Zariski open subset Z of X' and a morphism $p : Z \to U$ into a curve U whose fibers are isomorphic to $\mathbf{C}^2$;

(3) each irreducible component of $X' \setminus Z$ is a UFD.

Then U is isomorphic to a Zariski open subset of $\mathbf{C}$ and p can be extended to a regular function on X'. Furthermore, X' is isomorphic to $\mathbf{C}^3$ and p is a variable.

The same conclusion remains true if we replace (1) and (3) by

(1') X' is smooth and $H_3(X') = 0$; [4]

(3') each irreducible component of $X' \setminus Z$ has at most isolated singularities.

In the case when conditions (1') and (2) hold but (3) does not, X' is an exotic algebraic structure on $\mathbf{C}^3$ (that is, X' is diffeomorphic to $\mathbf{R}^6$ as a real manifold but not isomorphic to $\mathbf{C}^3$) with a nontrivial Makar-Limanov invariant.

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[1]In fact, P. Russell indicated to the author that the "hard-case" of the Linearization conjecture is equivalent to this theorem. This equivalence can be extracted from [KoRu1].

[2] If K is the field of fractions of $\mathbf{C}[p]$ then this means that the ring $\mathbf{C}[x, y, z] \otimes_{\mathbf{C}[p]} K$ is isomorphic to the polynomial ring in two variables over K.

[3]An affine algebraic variety is called a UFD if its algebra of regular functions is a UFD

[4]All homology groups which we consider in this paper have $\mathbf{Z}$-coefficients.

The Makar-Limanov invariant was introduced in [M-L1], [KaM-L1] (see also [KaM-L2], [Za], and [De]). For a reduced irreducible affine algebraic variety X' (and X' from the Main Theorem is reduced and irreducible since it is a UFD) this invariant is the subalgebra $\mathrm{ML}(X')$ [5] of the algebra of regular functions $\mathbf{C}[X']$ on X' that consists of all functions which are invariant under any regular $\mathbf{C}_+$-action on X'. If $\mathrm{ML}(X')$ coincides with the ring of constants then we say that it is trivial. This is so, for instance, when X' is isomorphic to $\mathbf{C}^n$.

The proof of the Main Theorem can be divided in three major steps. The first step is the following strengthened version of the theorem of Miyanishi [Miy1] (which is essentially based on [Sa2])

Lemma I. *Let X' be an affine algebraic variety of dimension 3 such that X' is a UFD, all invertible functions on X' are constants, and*

(1) the Euler characteristics of X' is $e(X') = 1$;

(2′) there exists a Zariski open subset Z of X' which is a $\mathbf{C}^2$ -cylinder over a curve U (i.e. Z is isomorphic to the $\mathbf{C}^2 \times U$);

(3) each irreducible component of $X' \setminus Z$ is a UFD.

Then X' is isomorphic to $\mathbf{C}^3$.

(4) Furthermore, the curve U is a Zariski open subset of $\mathbf{C}$, the natural projection from Z to U can be extended to a regular function on X' , and this function is a variable. [6]

The statement of this Lemma remains true if conditions (1) and (3) are replaced by

(1′) X' is smooth and $H_3(X') = 0$;

(3′) each irreducible component of $X' \setminus Z$ has at most isolated singularities.

We give a new proof of this theorem based on the notion of affine modifications which was studied in [KaZa1]. In brief the idea of the proof is as follows.

An affine modification is just a birational morphism of reduced irreducible complex affine algebraic varieties $\sigma : X' \to X$. The restriction of such a morphism to the complement of the exceptional divisor E of X' is an isomorphism between $X' \setminus E$ and $X \setminus D$ where D is a divisor of X. The image $C_0 = \sigma(E)$ is called the geometrical center of modification. For some affine modifications (which are called below affine cylindrical modifications) E is isomorphic to the direct product $\mathbf{C}^k \times C_0$

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[5]In papers of Makar-Limanov it is denoted by $\mathrm{AK}(X')$.

[6]In the original formulation of Miyanishi the statement (4) was absent but it was, obviously, known to Miyanishi. The last statement of this Lemma was also absent in the paper of Miyanishi.

(where $k = \operatorname{codim}{}_X C_0 - 1$) which enables us to compare the topology of X' and X. Cylindrical modifications may not preserve some features of X (for instance, normality) and their geometrical centers are not always closed. Therefore, we introduce a subset of so-called basic modification (in the set of all cylindrical modification) which enable us to control geometrical changes more effectively. We show that under the assumption of the Miyanishi theorem X' is an affine modification of $X = \mathbf{C}^3$ and the divisor D is the union of a finite number of parallel affine planes in $\mathbf{C}^3$. Then the problem will be reduced to the case when D consists of one plane only. One of the central facts for the first step is Theorem 2.3 which says that σ is the composition $\sigma_1 \circ \cdots \circ \sigma_m$ where each $\sigma_i : X_i \to X_{i-1}$ ($X' = X_m$ and $X = X_0$) is a basic modification. If $m = 1$ and C is either a point or a straight line in the plane D then it is easy to check that X' is isomorphic to $\mathbf{C}^3$ and the other statements of the Miyanishi theorem hold. In the general case of $m > 1$, using the control over topology, one can show that the center of σ_1 is either a point or a curve in D which is isomorphic to $\mathbf{C}$. If the center is such a curve then it can be viewed as a straight line by the Abhyankar-Moh-Suzuki theorem whence X_1 is isomorphic to $\mathbf{C}^3$. Now the induction by m implies the Miyanishi theorem.

In the second step we prove

Lemma II. *Let X' be an affine algebraic variety of dimension 3 such that X' is a UFD, all invertible functions on X' are constants, and let the assumption (1') and (2') of Lemma I hold, but (3) does not. Then X' is an exotic algebraic structure on $\mathbf{C}^3$ with a nontrivial Makar-Limanov invariant.*

Our proof of the Miyanishi theorem is longer than the original one but it has some advantage besides a slightly stronger formulation. It helps us to cope with the second step. Namely, under the assumption of Lemma II X' is still an affine modification of $X = \mathbf{C}^3$, σ is still a composition of basic modifications, and one can reduce the problem to the case when D is a coordinate plane. It can be shown that the geometrical center of modification is either a point or an irreducible contractible curve in D. Besides the Abhyankar-Moh-Suzuki theorem we have another remarkable fact in dimension 2 – the Lin-Zaidenberg theorem [LiZa] says that such a curve is given by $x^n = y^m$ in a suitable coordinate system where n and m are relatively prime. This allows us to present explicitly a system of polynomial equations in some Euclidean space $\mathbf{C}^N$ whose zero set is X'. Here we use the fact that basic modifications of Cohen-Macaulay varieties are Davis modifications which were introduced in [KaZa1]

and which fit perfectly the aim of presenting explicitly the result of a modification as a closed affine subvariety of a Euclidean space. This explicit presentation of X' as a subvariety of $\mathbf{C}^N$ enables us to compute the Makar-Limanov invariant of X', using the technique from [KaM-L1], [KaM-L2]. If condition ($3b'$) holds and X' is smooth then the Makar-Limanov invariant of X' is non-trivial whence X' is not isomorphic to $\mathbf{C}^3$. On the other hand we show that X' is contractible and, therefore, it is diffeomorphic to $\mathbf{R}^6$ by the Dimca-Ramanujam theorem [Di], [ChDi] which concludes the second step.

Third step is

Lemma III. *Let $q : Z \to U$ be a morphism of an affine algebraic variety Z into a curve U such that every fiber of q is isomorphic to $\mathbf{C}^2$. Then there exists a Zariski open subset U^* of U such that for $Z^* = q^{-1}(U^*)$ and $r = q|_{Z^*}$ the morphism $r : Z^* \to U^*$ is a $\mathbf{C}^2$-cylinder over U^* (that is, there exists an isomorphism $\varphi : Z^* \to \mathbf{C}^2 \times U^*$ for which the composition of the projection to the second factor and φ coincides with r).*

The proof of Lemma III will be the content of the next joint paper of the author and M. Zaidenberg [KaZa2].

The combination of Lemmas I, II, and III implies Main Theorem.

It is our pleasure to thank M. Zaidenberg for his suggestion to check Lemma II and many fruitful discussions. Actually, the idea of this paper arose during the joint work of the author and M. Zaidenberg on [KaZa1]. Later M. Zaidenberg decided not to participate in the project due to other obligations and the author had to finish it alone.

It is also our pleasure to thank I. Dolgachev whose consultations were very useful for the author.

2. Affine Modifications

2.1. **Notation and Terminology.** In this subsection we present central definitions and notation which will be used in the rest of the paper. The ground field in this paper will always be the field of complex numbers $\mathbf{C}$. But it should be noted that all facts of this section hold for every field of characteristics zero with the exception of the results where the homology or fundamental groups are mentioned.

Definition 2.1. We remind that an affine domain A over $\mathbf{C}$ is just the algebra of regular functions $\mathbf{C}[X]$ on a reduced irreducible complex affine algebraic variety X. Let I be an ideal in A and $f \in A \setminus \{0\}$. By the affine modification of A with locus (I, f) we mean the algebra $A' := A[I/f]$ together with the natural embedding $A \hookrightarrow A'$. That is, if $b_0, \ldots, b_s$ are generators of I then A' is the subalgebra of the field $\mathrm{Frac}(A)$ of fractions of A which contains A and which is generated over A by the elements $b_0/f, \ldots, b_s/f$. It can be easily checked [KaZa1] that A' is an affine domain provided A is. Hence the spectrum of A' is an affine algebraic variety X' and the natural embedding $A \hookrightarrow A'$ generates a morphism $\sigma : X' \to X$. Sometimes we refer to σ as an affine modification or we say that X' is an affine modification of X. The reduction D (resp. E) of the divisor $f^{-1}(0) \subset X$ (resp. $(f \circ \sigma)^{-1}(0) \subset X'$) will be called the divisor (resp. the exceptional divisor) of the modification. The subvariety of X defined by the ideal I (and sometimes the ideal I itself) will be called the center of the modification, its reduction (which coincides, of course, with the zero set of I in X) will be called the reduced center of the modification, and $\sigma(E)$ will be called the geometrical center of modification. [7]

Remark 2.1. (1) If A' is as above and $f \notin I$ then consider the ideal $J = \{I, f\}$ in A generated by I and f. Clearly, $A' = A[J/f]$ whence we shall suppose that $f \in I$ in the rest of the paper.

(2) It is easy to produce examples (and some of them appear below) which show that the center, the reduced center, and the closure of the geometrical center of a modification may be different but in all cases the geometrical center is contained in the reduced center. Indeed, otherwise one can choose an element $a \in A$ which vanishes identically on the latter but not on the former. By Nullstellensatz $a^n \in I$ for some natural n. On the other hand a^n/f takes on the ∞-value on X' whence this function cannot be regular on X'. Contradiction.

Definition 2.2. Let $p : Y \to Z$ be a morphism of algebraic varieties We say that p is a $\mathbf{C}^s$-cylinder over Z if there exists an isomorphism $\varphi : Y \to \mathbf{C}^s \times Z$ such that $p \circ \varphi^{-1}$ is the projection to the second factor.

Definition 2.3. Let $\sigma(E)$ be the geometrical center of the affine modification σ from Definition 2.1. Suppose that $\sigma(E)$ is not just a constructive set but an algebraic

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[7]Affine modifications appeared under different names in several papers to which the author and M. Zaidenberg did not pay sufficient attention while we were writing [KaZa1]. In particular, we missed at that time the notion of Néron's blowing-ups [Né] which appeared as early as in 1964 (see also [Ar], [WaWe]).

variety of pure dimension. We say that σ is a cylindrical modification of rank s if $\sigma|_E : E \to \sigma(E)$ is a $\mathbf{C}^s$-cylinder where $s+1$ is the codimension of the geometrical center in X.

It is useful to know when the geometrical center of a cylindrical modification coincides with the reduced center and, in particular, is closed (then we can better control the change of topology under the modification). Here is the definition of semi-basic modifications which are cylindrical and whose geometrical centers coincide with the reduced ones (see Proposition 2.7 below).

Definition 2.4. Let $b_0, \ldots, b_s$ be a sequence in an affine domain $A = \mathbf{C}[X]$ which generates an ideal I. We say that this sequence is semi-regular if the height of I is $s+1$ (or, equivalently, the zero set of I in X is a subvariety of pure codimension $s+1$). If in addition $b_0 = f$ then the affine modification $A \hookrightarrow A'$ with locus (I, f) will be called semi-basic of rank s, and $b_0, \ldots, b_s$ will be called a representative system of generators for this modification.

We shall see that semi-basic modifications preserve Cohen-Macaulay rings but they do not preserve normality. For this purpose we need to consider a more narrow class of modifications.

Definition 2.5. Let the notation of Definition 2.4 hold. A semi-regular sequence $b_0, b_1, \ldots, b_s$ is called an almost complete intersection provided the following two conditions hold

(i) none of the irreducible components of the set C of the common zeros of I in X is contained in the singularities of X;

(ii) for every irreducible component C_i of C there exists its Zariski open subset $C_i^0 \subset \operatorname{reg} X$ which is a complete intersection given by $b_0 = \cdots = b_s = 0$ (in a neighborhood of C_i^0). That is, the gradients of $b_0, \ldots, b_s$ are linearly independent at generic points of each irreducible component of C.

If in addition $b_0 = f$ then the affine modification $A \hookrightarrow A'$ will be called basic of rank s, and again $b_0, \ldots, b_s$ will be called a representative system of generators.

We shall need also affine modifications which are cylindrical (resp. semi-basic, basic) only locally. These notions will appear in the next subsection.

Convention 2.1. Further in this paper X and X' will **always** be reduced irreducible affine algebraic varieties. The algebra of regular functions of an affine algebraic variety Y will be $\mathbf{C}[Y]$. We put $A = \mathbf{C}[X]$ and $A' = \mathbf{C}[X']$, that is, A and A' will **always** be the affine domains that correspond to the affine varieties X and X'

respectively. Furthermore, we suppose that the notation $A \hookrightarrow A'$ is fixed throughout the paper. It will **always** mean an affine modification with locus (I, f). The corresponding morphism of the algebraic varieties will **always** be denoted by $\sigma : X' \to X$. The divisor, the exceptional divisor, and the reduced center of the modification will be **always** denoted by D, E, and C respectively.

We shall also use the following notation in the rest of this section: if Y is an affine algebraic variety and $B = \mathbf{C}[Y]$ then for every closed algebraic subvariety Z of Y the defining ideal of Z in B will be denoted by $\mathcal{I}_B(Z)$. For every ideal J in $\mathbf{C}[Y]$ we denote by $\mathcal{V}_Y(J)$ the zero set of this ideal in Y.

2.2. **General Facts about Affine Modifications.** We shall list first several useful facts from [KaZa1].

Theorem 2.1. (1) [KaZa1, Lemmas 1.1 and 1.2] *Let $A \hookrightarrow A'$ be an affine modification. Then the fields of fractions $\mathrm{Frac}(A)$ and $\mathrm{Frac}(A')$ coincide, i.e. σ is a birational morphism. The restriction of σ to $X' \setminus E$ is an isomorphism between $X' \setminus E$ and $X \setminus D$.*

(2) [KaZa1, Th. 1.1] *Every birational morphism $X' \to X$ of affine algebraic varieties is a modification. That is, there exist an ideal $I \subset A$ and $f \in A$ such that $A' = A[I/f]$.*

(3) [KaZa1, Prop. 1.2] *Let $f = f_1 f_2$ and $A' = A[I/f]$. Then $A' = A^1[I^2/f_2]$ where $A^1 = A[I_1/f_1]$, I_1 is the ideal in A generated by I and f_1, and I^2 is the ideal in A^1 generated by I/f_1.*

(4) [KaZa1, Prop. 3.1 and Th. 3.1] *Let $A \hookrightarrow A'$ be an affine modification. Suppose that E and D are topological manifolds and they have the same number of irreducible components. Furthermore, for every such component E_0 of E there exists a unique component D_0 of D for which $E_0 = \sigma^*(D_0)$ and $\sigma(E_0) \cap \mathrm{reg}\, D_0 \neq \emptyset$. Suppose also that $\sigma|_E : E \to D$ generates an isomorphism of the homology groups. Then $\sigma : X' \to X$ generates isomorphisms of the homology and fundamental groups.*

(5) [KaZa1, Cor. 2.1] *Let $X_1 = \mathrm{spec}\, A_1$ be an irreducible closed subvariety of X. Let the ideal $I_1 \subset A_1$ consist of the restrictions to X_1 of the elements of I. Suppose that $f_1 := f \,|\, X_1 \neq 0$. Consider the modification $A_1 \hookrightarrow A_1'$ with locus (I_1, f_1) and the corresponding morphism of algebraic varieties $\sigma_1 : X_1' \to X_1$ where $X_1' = \mathrm{spec}\, A_1'$. Then there is a unique closed embedding $i' : X_1' \hookrightarrow X'$ making the following diagram commutative:*

$$\begin{array}{ccc} X_1' & \stackrel{i'}{\hookrightarrow} & X' \\ \downarrow{\scriptstyle\sigma_1} & & \downarrow{\scriptstyle\sigma} \\ X_1 & \stackrel{i}{\hookrightarrow} & X \end{array}$$

where $i : X_1 \hookrightarrow X$ is the natural embedding. In particular, affine modifications commute with direct products.

We need to discuss the behavior of affine modifications under localizations (this should have been done in [KaZa1]). Let S be a multiplicative subset of A and $S^{-1}A$ (resp. $S^{-1}A'$) be the localization of A (resp A') with respect to S. Every ideal I in A generates an ideal $S^{-1}I$ in $S^{-1}A$. The intersection of $S^{-1}I$ with A is an ideal $S(I)$ which contains I. The following fact is an immediate consequence of the definitions of affine modifications and localizations.

Proposition 2.1. *In the notation above we have* $S^{-1}A' = (S^{-1}A)[S^{-1}I/f] = (S^{-1}A)[S^{-1}(S(I))/f]$. *That is, localizations and affine modifications commute.*

Definition 2.6. Suppose that B is a localization of an affine domain, J is an ideal in B and $g \in B \setminus \{0\}$. By the local affine modification of B with locus (J, g) we mean the algebra $B' := B[J/g]$ together with the natural embedding $B \hookrightarrow B'$. By Proposition 2.1 B' is also a localization of an affine domain. Hence the spectrum of B (resp. B') is a (germ of an) affine algebraic variety Y (resp. Y') and the natural embedding $B \hookrightarrow B'$ generates a morphism $\delta : Y' \to Y$. We can define now the divisor, the exceptional divisor, the (reduced, geometrical) center of this local modification exactly in the same manner we did for affine modifications.

Remark 2.2. By Proposition 2.1 each local affine modification $B \hookrightarrow B'$ is just a localization of an affine modification $A \hookrightarrow A'$ which respect to a multiplicative system $S \subset A$. This implies that Theorem 2.1 (1)-(3) and (5) hold for local affine modifications as well. Similarly, some facts below (including Theorem 2.3) can be easily reformulated for the local case.

Definition 2.7. (1) A local affine modification $B \hookrightarrow B'$ is called cylindrical (resp. semi-basic, basic) if an affine modification $A \hookrightarrow A'$ from Remark 2.2 can be chosen cylindrical (resp. semi-basic, basic).

(2) Let $A \hookrightarrow A'$ be an affine modification. Suppose that M is a maximal ideal of A. Recall that the localization A_M of A near M is the localization of A with respect to the multiplicative system $S = A \setminus M$. Let I_M denote the ideal generated in

A_M by I. We say that this affine modification is locally cylindrical (resp. semi-basic, basic) if for every point of the geometrical center $\sigma(E)$ and the maximal ideal M, that vanishes at this point, the local affine modification $A_M \hookrightarrow A_M[I_M/f] = S^{-1}A'$ is cylindrical (resp. semi-basic, basic).

As we mentioned before there exist affine modifications whose reduced center is different from the closure of the geometrical one. That is, for such a modification the natural projection $E \to C$ is not dominant where E is the exceptional divisor of the modification and C is its reduced center. (Consider, for instance, $A' = A[\{f\}/f^2]$. Here E is empty and C is not.) In order to have control over the change of topology under an affine modification we need to understand when the reduced center coincides with the closure of the geometrical one. This requires the notion of the largest ideal of a modification $A' = A[I/f]$ which was introduced in [KaZa1].

Definition 2.8. The ideal $K = \{a \in A \mid a/f \in A'\}$ in A is called the f-largest ideal of the modification $A \hookrightarrow A'$. Clearly, $I \subset K$ and $A' = A[K/f]$. When A and A' are fixed we denote this largest ideal K by I_f.

Proposition 2.2. Let $A \hookrightarrow A'$ be an affine modification such that $I = I_f$. Then the reduced center of the modification coincides with the closure of the geometrical one. (This means that for every component C_0 of the reduced center, $\sigma^{-1}(C_0)$ is a hypersurface in X' whose image is dense in C_0.)

Proof. Assume that $\sigma^{-1}(U)$ is empty for some Zariski open subset U of C_0. Choose a regular function $a \in A$ so that a vanishes on each component of the reduced center except for C_0 but we require that a vanishes also on $C_0 \setminus U$. Then $a' = a \circ \sigma$ vanishes on the exceptional divisor E. Note that the zeros of $f' := f \circ \sigma$ on X' coincide with E. By Nullstellensatz for some $n > 0$ the element $(a')^n$ is divisible by f' in A'. Since $(a')^n/f' = a^n/f \in A'$ we have $a^n \in I_f$ which shows that the reduced center does not contain C_0. Contradiction. $\quad\square$

Remark 2.3. As a consequence of Proposition 2.2 we see that for $I = I_f$ the number of irreducible components of the exceptional divisor E is at least the same as the number of irreducible components of the reduced center of the modification. There is a better estimate of the number of irreducible components of E (which will not be used further). Namely, it can be shown that in the case of a normal affine domain A this number is greater than or equal to the number of ideals in a minimal primary decomposition of I.

Here is another property of the largest ideals of affine modifications which will be used again and again in this section.

Proposition 2.3. *Let* $g \in A \setminus \{0\}$ *and* $f = g^n$ *for a natural* n *. Suppose that the defining ideal* $\mathcal{I}_{A'}(E)$ *(see the end of subsection 2.1 for notation) of the exceptional divisor* E *of the modification* $A \hookrightarrow A'$ *coincides with the principal ideal in* A' *generated by* g *. Then* $(\mathcal{I}_A(C))^n \subset I_f$ *(in particular, for* $n = 1$ *we have* $\mathcal{I}_A(C) = I_f$ *).*

Furthermore, for every ideal J *in* A *which is contained in* $\mathcal{I}_A(C)$ *the algebra* $A_1 := A[J/g]$ *is contained in* A' *.*

Proof. Note that for every $a \in (\mathcal{I}_A(C))^n$ we have $a \in (\mathcal{I}_{A'}(E))^n$ whence $a/f \in A'$. By Definition 2.8 $a \in I_f$ which is the first statement. This implies that $g^{n-1}J \subset I_f$. Hence $A_1 = A[g^{n-1}J/f] \subset A[I_f/f] = A'$. □

Remark 2.4. Suppose that A_1 be an affine domain such that $A \subset A_1 \subset A'$ (for instance, A_1 is from Proposition 2.3).

(1) Since every generator of A_1 over A is of form b/f^m where $b \in I^m$ we see that there exists an ideal L_1 in A such that $A_1 = A[L_1/f^m]$ for some $m \geq 0$.

(2) Furthermore there exists an ideal K_1 in A_1 such that $A' = A_1[K_1/f]$ (it is enough to consider the ideal generated by I in A_1 as K_1). Hence the modification $A_1 \hookrightarrow A'$ with locus (K_1, f) has the same exceptional divisor E as the modification $A \hookrightarrow A'$ and the divisor of $A_1 \hookrightarrow A'$ coincides with the exceptional divisor of the affine modification $A \hookrightarrow A_1$ with locus (L_1, f^m) . In fact, under the assumption of Proposition 2.3 we can make a stronger claim which will help us later to show that the number of factors in the desired decomposition of an affine modification into basic modifications is finite.

Corollary 2.1. (cf. [WaWe, Prop. 1.2]) *Suppose that* $J = \mathcal{I}_A(C)$ *in Proposition 2.3. Then there exists an ideal* K_1 *in* A_1 *such that* $A' = A_1[K_1/g^{n-1}]$ *. That is,* $A_1 \hookrightarrow A'$ *may be viewed as an affine modification with locus* (K_1, g^{n-1}) *.*

Proof. Let $b_0 = g^n, b_1, \ldots, b_s$ be generators of I . Note that $b_i/g \in A_1$ for every i . The ideal K_1 in A_1 generated by $g^{n-1}, b_1/g, \ldots, b_s/g$ is the desired ideal. □

We remind that we are planing to show that X' from the Main theorem is a modification of $\mathbf{C}^3$. Since both of these threefolds are UFDs we need to have a closer look at affine modifications $A \hookrightarrow A'$ of UFDs. We shall see that the assumption of Proposition 2.3 on $\mathcal{I}_{A'}(E)$ is automatically true in this case.

Proposition 2.4. *Let* $A \hookrightarrow A'$ *be an affine modification, let* A *be a UFD, and let the following conditions hold*

(i) $f = g^n$ where $g \in A$ is irreducible;

(ii) the closure of the geometrical center of the modification coincides with the reduced center (by Proposition 2.2 this is so, for instance, when I is the f-largest ideal I_f of the modification);

(iii) $A' \neq A[1/f]$ or, equivalently, the exceptional divisor E of the modification is not empty.

Then

(1) g is irreducible as an element of A' ;

(2) if A' is also a UFD then E and, therefore, C are irreducible, and $\mathcal{I}_{A'}(E)$ coincides with the principal ideal generated by g.

Proof. Let $g^k = a'b'$ where $a' = a/f^l, b' = b/f^m, a \in I^l$, and $b \in I^m$. Hence $g^{k+nl+nm} = ab$ in A . Since A is a UFD we have $a = ug^s$ and $b = vg^r$ where $s + r = k + nl + nm$ and u, v are units. If $s < nl$ then $a' = u/f^{nl-s}$ whence $1/f \in A'$. This contradicts (iii). Thus $s \geq nl$ and, similarly, $r \geq nm$. Hence $a' = ug^{s-nl}$ and $b' = vg^{r-nm}$ are elements of A which implies (1).

Assume that $E = E_1 \cup E_2$ where E_1, E_2 are effective divisors of X' (where X' is as in Definition 2.1) without common irreducible components. If A' is a UFD E_k is the zero set of some $a'_k \in A'$. Hence $g = u(a'_1)^{n_1}(a'_2)^{n_2}$ where u is a unit. Since a'_k is not a unit this contradicts to the fact that g is irreducible in A' whence we have (2). $\square$

Remark 2.5. (1) Remark 2.3 implies that if A and A' in Proposition 2.4 are UFDs then the ideal I_f is primary.

(2) The following generalization of Proposition 2.4 will appear in a coming paper [KaVeZa] of M. Zaidenberg, S. Venereau, and the author. Let $A \hookrightarrow A'$ be an affine modification such that A and A' are UFDs which have the same units. Then the numbers of irreducible components in D and E are the same.

Example 2.1. (1) Lemma 2.3 implies that if A is a UFD and E is not irreducible then A' is not a UFD. Consider, for instance, $A = \mathbf{C}[x, y]$ and $f = x$. Let I be generated by x and $y^2 - y$. Then X' is the surface in $\mathbf{C}^3$ (with coordinates x, y, z) given by $xz = y^2 - y$. This is a so-called Danielevski surface. These surfaces are not UFDs. The exceptional divisor in this case consists of two components $x = y = 0$ and $x = y - 1 = 0$.

(2) In the case when E is irreducible A' may be not a UFD anyway, if, say, $\mathcal{I}_{A'}(E)$ is not a principal ideal. Let A and f be as in the first example, and let I be generated by x and y^2 . Then X' is the surface in $\mathbf{C}^3$ given by $xz = y^2$. It is not a UFD.

In general the units of A and A' differ (consider, for instance, $A' = A[1/f]$) but under some mild assumption this is not the case (in particular, in Proposition 2.4 A and A' have the same units).

Proposition 2.5. *Let $A \hookrightarrow A'$ be an affine modification. Suppose that for every natural k each irreducible divisor g of f^k in A is not a unit in A' or, equivalently, the set $(g \circ \sigma)^{-1}(0)$ is not empty. (When A is a UFD it is enough, of course, to consider the irreducible divisors of f only.) Then the units of A' and A are the same.*

Proof. Note that A' is a subalgebra of $A[1/f]$. Thus its units are also units of $A[1/f]$. The units of the last algebra are the products of irreducible divisors of f^k and the units of A. By the assumption these divisors are not invertible functions on X' whence the units of A' coincide with the units of A. $\qquad\square$

Proposition 2.6. *Let I_j be an ideal in A for $j = 1, \ldots, k$, and let $f_j \in I_j \setminus \{0\}$. Suppose that $f = f_1 \cdots f_k$ and $I = (f/f_1)I_1 + \ldots + (f/f_k)I_k$. Let $A_j = A[I_j/f_j]$ and let $\sigma_j : X_j \to X$ be the morphism of affine algebraic varieties associated with the affine modification $A \hookrightarrow A_j$ with locus (I_j, f_j). Suppose that E_j is the exceptional divisor of this modification. These morphisms define the affine variety $Y = X_1 \times_X X_2 \times_X \cdots \times_X X_k$ and its subvariety $Y^* = (X_1 \setminus E_1) \times_X \cdots \times_X (X_k \setminus E_k)$.*

(1) The variety X' is isomorphic to the closure $\bar{Y}^*$ of Y^* in Y and under this isomorphism σ coincides with the restriction of the natural projection $\tau : Y \to X$ to $\bar{Y}^*$.

(2) If f_j and f_l have no common zeros on X for every pair $j \neq l$ then X' is isomorphic to Y.

Proof. Let D_j be the zero locus of f_j on X. Then $D = \bigcup_{j=1}^{k} D_j$. Since the restriction of σ_j to $X_j \setminus E_j$ is an isomorphism between $X_j \setminus E_j$ and $X \setminus D_j$ we see that Y^* is isomorphic to $X \setminus D$. In particular, the natural projection $\bar{Y}^* \to X$ generates an isomorphism of the fraction fields of the algebras $B := \mathbf{C}[\bar{Y}^*]$ and A. That is, $B \subset \mathrm{Frac}\,(A)$. The natural projection $\bar{Y}^* \to X_j$ enables us to treat A_j as a subalgebra of B. Furthermore, since $\bar{Y}^*$ is the subvariety of Y we see that B is generated by these subalgebras $A_1, \ldots, A_k$. It remains to note that A' is also generated by $A_1, \ldots, A_k$ since $I = (f/f_1)I_1 + \ldots + (f/f_k)I_k$. Hence $A' = B$ and X' is naturally isomorphic to $\bar{Y}^*$ which yields (1).

For the second statement it suffices to prove that Y is irreducible. Assume that Y has an irreducible component Y_1 different from $\bar{Y}^*$. Then the image of this component under the projection $\tau : Y \to X$ must be contained in D since the restriction of τ^{-1}

to $X \setminus D$ is an isomorphism between $X \setminus D$ and Y^*. We can suppose that this image is contained in D_1. Put $T = \bigcup_{j=2}^{k} D_j$ and consider $\theta : Y \setminus \tau^{-1}(T) \to X \setminus T$ where θ is the restriction of τ. Since for $j \geq 2$ the restriction of σ_j to $X_j \setminus \sigma_j^{-1}(T)$ is an isomorphism between this variety and $X \setminus T$ we see that $Y \setminus \tau^{-1}(T)$ is isomorphic to $X_1 \setminus \sigma_1^{-1}(T)$ and θ coincides with the restriction of σ_1 to $X_1 \setminus \sigma_1^{-1}(T)$ under this isomorphism. Thus $\sigma_1^{-1}(X \setminus T) \simeq \theta^{-1}(X \setminus T) \simeq \tau^{-1}(X \setminus T)$. Note that T does not meet D_1 by the assumption of this Proposition. Hence $\tau^{-1}(X \setminus T)$ contains Y_1 and, therefore, it is not irreducible. On the hand $\sigma_1^{-1}(X \setminus T) \subset X_1$ is irreducible. This contradiction concludes (2). $\qquad \square$

Remark 2.6. Let us discuss the coordinate meaning of Proposition 2.6 (2). Suppose for simplicity that $k = 2$. Let X be a closed affine subvariety of $\mathbf{C}^n$ with a coordinate system $\bar{x}$ and let X_j be a closed affine subvariety of $\mathbf{C}^{n_j}$ with a coordinate system $(\bar{x}, \bar{z}_j)$. That is, $\mathbf{C}^{n_j}$ contains the above sample of $\mathbf{C}^n$ as a coordinate n-plane. Suppose that X_j coincides with the zeros of a polynomial system of equations $P_j(\bar{x}, \bar{z}_j) = 0$ and σ_j can be identified with the restriction of the natural projection $\mathbf{C}^{n_j} \to \mathbf{C}^n$. Consider the space $\mathbf{C}^{n_1+n_2-n}$ with coordinates $(\bar{x}, \bar{z}_1, \bar{z}_2)$. Then Proposition 2.6 (2) implies that the set of zeros of the system $P_1(\bar{x}, \bar{z}_1) = P_2(\bar{x}, \bar{z}_2) = 0$ in this space is isomorphic to X'.

2.3. **Semi-basic Modifications.** In general the topologies of the exceptional divisor E and the reduced center C of an affine modification are not related very well even in the case when the reduced center coincides with the closure of the geometrical one. The natural projection $E \to C$ may not be surjective and, furthermore, its generic fibers may be not connected.

Example 2.2. Consider A equal to the ring $\mathbf{C}[x, y, z]$ of polynomials in three variables. Let $f = x$ and the ideal I be generated by x and y. Then $A' = A[I/f] = \mathbf{C}[x, u, v]$ where $y = xu, z = v$. In this case C is the z-axis and E is the uv-plane. Let Γ be a closed curve in the uv-plane whose projection to the v-axis is dominant but neither surjective nor injective. Suppose that $g(u, v) = 0$ is the defining equation of this curve in the uv-plane. Consider the ideal J in A' generated by f and g, and put $A'' = A'[J/x]$. The sequence f, g is regular in A', and we shall see later in this subsection (Proposition 2.7) that its exceptional divisor F is isomorphic to $\Gamma \times \mathbf{C}$. Note that the natural embedding of A into A'' is also a modification by Theorem 2.1 (2) which has the same exceptional divisor F. The projection $F \to C$

14

is the composition of the projections $F \to \Gamma$ and $\Gamma \to C$. This yields the desired example.

This example suggests also an approach to what should be done in order to track the change of topology. We shall try to present an affine modification $A \hookrightarrow A'$ as a composition of basic modifications. If A_1 is an affine domain such that $A \subset A_1 \subset A'$ then this modification is the composition of $A \hookrightarrow A_1$ and $A_1 \hookrightarrow A'$, where the last two embeddings are affine modifications by Theorem 2.1 (2) (see also Remark 2.4). When $f = g^n$ then Proposition 2.3 suggests to look for A_1 in the form $A_1 = A[J/g]$ where $J \subset \mathcal{I}_A(C)$. Our first aim in this subsection is to show that J can be chosen so that the affine modification $A \hookrightarrow A_1$ with locus (J, g) is semi-basic, and, under some additional assumption, even basic.

Lemma 2.1. *Let C be a closed reduced subvariety of X of codimension $s+1$ and let $I = \mathcal{I}_A(C)$. Suppose that $f \in I \setminus \{0\}$.*

(1) Then one can choose a semi-regular sequence $f = b_0, \ldots, b_s$ whose elements are contained in I.

(2) Let this sequence generate an ideal J. If none of the irreducible components of C and none of the irreducible components of the zero divisor of f is contained in the singularities $\operatorname{sing} X$ of X then the sequence above can be chosen so that none of the irreducible components of $\mathcal{V}_X(J)$ is contained in $\operatorname{sing} X$.

(3) If the assumption of (2) holds and the zero multiplicity of f at generic points of each irreducible component of C is 1, then the sequence $b_0 = f, b_1, \ldots, b_s$ can be chosen so that it is an almost complete intersection.

(4) There exists a finite-dimensional subspace S of I such that (1)-(3) are true when $b_1, \ldots, b_s$ are generic points of any finite-dimensional subspace of I which contains S.

Proof. Suppose that X is a closed subvariety of $\mathbf{C}^n$ and $\bar{x} = (x_1, \ldots, x_n)$ is a coordinate system on $\mathbf{C}^n$. Let $g_0, g_1, \ldots, g_r$ be generators of I and let f be one of them (say, $f = g_0$). Consider the set S_m of $(m+1)$-tuples $\bar{b}_m = (b_0 = f, b_1, \ldots, b_m)$ such that each $b_i \, (i \geq 1)$ is of the form $\sum_{j=0}^r l_j(\bar{x}) g_j$ where each $l_j(\bar{x})$ is a linear polynomial on $\mathbf{C}^n$. Let $W(\bar{b}_m)$ be the set of common zeros of $b_0 = f, b_1, \ldots, b_m$. We want to show that for $m \leq s$ the statements (1)-(3) are true with s replaced by m. By the assumption this is so when $m = 0$. Suppose that this is true for $m < s$ and show that (1)-(3) holds for $m + 1$. For every point $o \in X \setminus C$ we can find g_j which does not vanish at o. Thus, taking, a linear combination of $g_0, g_1, \ldots, g_r$ as b_{m+1} we can suppose that b_{m+1} does not vanish at generic points of every irreducible

component of $W(\bar{b}_m)$. Hence the sequence $b_0, \ldots, b_{m+1}$ generates an ideal J_{m+1} such that $\mathcal{V}_X(J_{m+1})$ has codimension $m+2$ in X.

If the assumption of (2) holds then $T = \operatorname{sing} X \cap W(\bar{b}_m)$ does not contain any irreducible component of $W(\bar{b}_m)$. We can assume that b_{m+1} does not vanish at generic points of every irreducible component of $T \setminus C$. Hence $\operatorname{sing} X \cap W(\bar{b}_{m+1})$ does not contain any irreducible component of $W(\bar{b}_{m+1})$ which is (2).

Since S_{m+1} can be viewed as an algebraic variety and $W(\bar{b}_{m+1})$ depends algebraically on $\bar{b}_{m+1}$ we see that the number of irreducible components of $W(\bar{b}_{m+1})$ at generic points of S_{m+1} is the same. Thus if we perturb b_{m+1} by an element of form $\sum_{j=0}^{r} l_j(\bar{x}) g_j$ we shall still have (1) and (2), i.e. (1) and (2) hold for generic $\bar{b}_{m+1} \in S_{m+1}$. Since the codimension of C is $s+1$ and $I = \mathcal{I}_A(C)$ for every generic point of C (which is a smooth point and which belongs to $\operatorname{reg} X$) we can always find $m+1$ elements among $g_1, \ldots, g_r$ such that the gradients of these elements and the gradient of $g_0 = f$ are linearly independent at this point. Thus one can suppose that the gradients of $b_0, \ldots, b_{m+1}$ are linearly independent at generic points of C. Consider a generic point $o \in W(\bar{b}_{m+1}) \setminus C$. One can suppose that $g_1(o) = 1$. Choose a linear polynomial $l_1(\bar{x})$ such that $l_1(o) = 0$. Then the gradients of $g_1 l_1$ and l_1 at o are the same. Therefore, perturbing b_{m+1} by a function of form $g_1 l_1$ one can suppose that the gradients of $b_0, \ldots, b_{m+1}$ are linearly independent at o whence they are linearly independent at generic points of the irreducible component of $W(\bar{b}_{m+1})$ that contains o. Since the number of irreducible components of $W(\bar{b}_{m+1})$ does not change in a neighborhood of a generic point $\bar{b}_{m+1}$ of S_{m+1} we can perturb b_{m+1} so that we have the linear independence of the gradients at generic points of each irreducible component of $W(\bar{b}_{m+1})$ which is (3).

For (4) it suffices to put $S = S_s$. $\qquad \square$

Proposition 2.7. *Suppose that $A \hookrightarrow A'$ is a semi-basic modification of rank $s > 0$. Then it is a cylindrical modification of rank s. Furthermore, the reduced and geometrical centers of this modification coincide.*

Proof. We shall begin with an example which was also described in [KaZa1]. Let J_0 be the maximal ideal in $\mathbf{C}^{[s+1]} = \mathbf{C}[x_0, x_1, \ldots, x_s]$ generated by all coordinates (i.e. it corresponds to the origin o in $\mathbf{C}^{s+1}$). Put $B_0 = \mathbf{C}^{[s+1]}[J_0/x_0]$ and consider the modification $\mathbf{C}^{[s+1]} \hookrightarrow B_0$ with locus (J_0, x_0). Then B_0 is isomorphic to $\mathbf{C}[x_0, y_1, \ldots, y_s]$ and $x_i = x_0 y_i$ for $i = 1, \ldots, s$. That is, $Z_0 := \operatorname{spec} B_0$ may be viewed as the subvariety of $\mathbf{C}^{2s+1}$ (whose coordinates are $x_0, x_1, \ldots, x_s, y_1, \ldots, y_s$) given by the system of equations $x_i - x_0 y_i = 0, i \geq 1$. Let $\rho : \mathbf{C}^{2s+1} \to \mathbf{C}^{s+1}$ be the natural projection to the first $s+1$ coordinates. Then our modification is nothing

16

but the restriction of ρ to Z_0. Its reduce and geometrical centers are o and the exceptional divisor $E_0 = \rho^{-1}(o) \simeq \mathbf{C}^s$.

Put $Z = \mathbf{C}^{s+1} \times X$ and $B = \mathbf{C}[Z]$. That is, $B = \mathbf{C}^{[s+1]} \otimes A = A^{[s+1]} = A[x_0, x_1, \ldots, x_s]$. Put $J = J_0 B$ and consider the modification $B \hookrightarrow B'$ with locus (J, x_0). Since modifications commute with direct products (see Theorem 2.1 (5)) we see that $Z' := \operatorname{spec} B' = Z_0 \times X$ and the above modification is the restriction δ to $Z' \subset \mathbf{C}^{2s+1} \times X$ of the natural projection $(\rho, \operatorname{id}) : \mathbf{C}^{2s+1} \times X \to \mathbf{C}^{s+1} \times X = Z$. In particular, its reduced and geometrical centers are $C^0 = o \times X$ and the exceptional divisor $E^0 = E_0 \times X$.

Let $b_0 = f, b_1, \ldots, b_s$ be a representative system of generators for $A \hookrightarrow A'$. In particular, this sequence generates I. Consider the embedding $i : X \hookrightarrow Z$ given by the system of equations $x_i - b_i = 0, i = 0, \ldots, s$. Then the restriction of J to X coincides with I. By Theorem 2.1 (5) we have the commutative diagram

$$
\begin{array}{ccc}
X' & \overset{i'}{\hookrightarrow} & Z' \\
\downarrow{\sigma} & & \downarrow{\delta} \\
X & \overset{i}{\hookrightarrow} & Z
\end{array}
$$

where $i' : X' \hookrightarrow Z'$ is a closed embedding. The reduced center of $\sigma : X' \to X$ coincides with $C = C^0 \cap i(X)$, and it is of codimension $s+1$ in X since σ is semi-basic. Since E is of codimension 1 in X' we see that each generic fiber F of $\sigma|_E : E \to \sigma(E) \subset C$ must be at least of dimension s. But F is contained in a fiber $F^0 \simeq \mathbf{C}^s$ of $\delta|_{E^0} : E^0 \to C^0$. Hence $\dim F = s$ and $\sigma(E)$ is dense in C. Furthermore, since i' is a closed embedding $F = F^0$ and $\sigma(E) = C$ whence E is a $\mathbf{C}^s$-cylinder over C and the reduced and geometrical centers of this modification coincide. $\qquad \square$

Example 2.3. Not every cylindrical modification is semi-basic (even locally). Consider the algebra A of regular functions on $X = \{xy = zt\} \subset \mathbf{C}^4_{x,y,z,t}$. Let $f = x^2$ and the ideal I be generated by x^2, yx, y^2, and z. The the reduced (and the geometrical) center C of the modification $A \hookrightarrow A'$ is the line $x = y = z = 0$. Clearly this modification is not semi-basic since this line cannot be given by the zeros of x^2 and one more regular function. One can present X' as the hypersurface $\{u = vt\} \subset \mathbf{C}^4_{x,u,v,t}$ where $xu = y$ and $x^2 v = z$ and the exceptional divisor E in X' coincides with the zeros of x. It is easy to see that the projection $E \to C$ is the cylinder.

2.4. **Davis modifications.** It is useful to compare semi-basic modifications with Davis modifications which were introduced in [KaZa1]

Theorem 2.2. ([Da], see also [Ei, Ex. 17.14]) *Let* $f = b_0, b_1, \ldots, b_s$ *be generators of an ideal* J *in a Noetherian domain* B *. Consider the surjective homomorphism*

$$\beta : B^{[s]} = B[y_1, \ldots, y_s] \longrightarrow B[J/f] = B[b_1/f, \ldots, b_s/f] \simeq B'$$

where $y_1, \ldots, y_s$ *are independent variables and* $\beta(y_i) = b_i/f$, $i = 1, \ldots, s$. *Denote by* J' *the ideal of the polynomial algebra* $B^{[s]}$ *generated by the elements* $L_1, \ldots, L_s \in \ker \beta$ *where* $L_i = f y_i - b_i$. *Then* $\ker \beta$ *coincides with* J' *iff* J' *is a prime ideal. The latter is true, for instance, if the system of generators* $b_0 = f, b_1, \ldots, b_s$ *of the ideal* J *is regular* [8].

Definition 2.9. Let B be (a localization of) an affine domain. When J' from Theorem 2.2 is prime the (local) affine modification $B \hookrightarrow B'$ with locus (J, f) is called Davis, and the sequence $b_0, b_1, \ldots, b_s$ is called a representative system of generators for this modification.

Remark 2.7. It is easy to see that every (local) affine Davis modification $B \hookrightarrow B'$ as above is cylindrical of rank s [KaZa1, Prop. 1.1 (c)]. The reduced and the geometrical centers coincide (but the center of the modification can still be different from the reduced center, see Examples 1.2 and 1.5 in [KaZa1]). In particular, the codimension of every irreducible component of the reduced center is $s + 1$ unless this center is empty. Hence this Davis modification is not only cylindrical but automatically semi-basic in the case of a non-empty reduced center.

Recall that if J is an ideal of an affine domain A and M is a maximal ideal in A then we denote the localization of A near M (i.e. the localization with respect to the multiplicative system $S = A \setminus M$) by A_M and the ideal generated by J in A_M by A_M.

Proposition 2.8. *Let* $A \hookrightarrow A'$ *be an affine modification, and let* $b_0 = f, b_1, \ldots, b_s$ *be a system of generators of* I *. If* M *is a maximal ideal in* A *then by Proposition 2.1* $A_M \hookrightarrow S^{-1}A'$ *is the local affine modification with locus* (I_M, f) *. Suppose that for every maximal ideal* M *this local modification is Davis and* $b_0, \ldots, b_s$ *is a representative system of generators in the ideal* I_M *. Then* $A \hookrightarrow A'$ *is a Davis modification.*

Proof. Let $A^{[s]} = A[y_1, \ldots, y_s]$ and let I' be the ideal in $A^{[s]}$ generated by $L_i = y_i f - b_i$, $i = 1, \ldots, s$. Put $Y = \mathbf{C}^s \times X$ (in particular, $A^{[s]} = \mathbf{C}[Y]$). Let Y_1 be

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[8]I.e. the ideal $(b_0, \ldots, b_s)$ is proper and for each $i = 1, \ldots, s$ the image of b_i is not a zero divisor in $B/(b_0, \ldots, b_{i-1})$.

the subvariety of Y defined by the ideal I' in $A^{[s]}$. We need to show that I' is prime, i.e. Y_1 is reduced irreducible. Choose a maximal ideal M' in $A^{[s]}$ which vanishes at a point $x' \in X'$. Let $x = \sigma(x')$ where $\sigma : X' \to X$ is the natural projection and let M be the maximal ideal of A that vanishes at x. Then $A \setminus M \subset A^{[s]} \setminus M'$ and $A^{[s]}_{M'}$ is a further localization of $S^{-1}A^{[s]}$. Since $A_M \hookrightarrow S^{-1}A'$ is a Davis modification and $b_0, \ldots, b_s$ is a representative system of generators of this modification by assumption, the ideal $S^{-1}I'$ is prime in $S^{-1}A^{[s]}$. But the localization $I'_{M'}$ of this ideal must be also prime. Hence the germ of Y_1 at x' is reduced irreducible.

If we want to claim the same about Y_1 we need to show that it is connected (for instance, irreducible). Note that $E_1 = Y_1 \cap f^{-1}(0) \simeq \mathbf{C}^s \times C$ where $C = \{b_0 = \cdots = b_s = 0\}$ is the reduced center of the modification. Since the localizations of our modification are Davis the codimension of irreducible component of C in X must be $s + 1$ by Remark 2.7. Hence $\dim E_1 = \dim X - 1$ unless E_1 is empty. By construction $Y_1 \setminus E_1$ is isomorphic to $X \setminus D$ and, therefore, irreducible. Furthermore, since the codimension of each irreducible component of Y_1 in Y is at most s (i.e. the dimension of such a component is at least $\dim X$), we see that the numbers of irreducible components of Y_1 and $Y_1 \setminus E_1$ are the same. Thus Y_1 is irreducible. $\square$

Another notion we have to use is Cohen-Macaulay rings. We send the readers for the definition and properties of these objects to [Ei] or [Ma]. We reverse first Remark 2.7 and show that every semi-basic modification of a Cohen-Macaulay affine domain is Davis.

Proposition 2.9. *Let A be Cohen-Macaulay and I the ideal generated by a semi-regular sequence $b_0 = f, b_1, \ldots, b_s$. Then the affine modification $A \hookrightarrow A'$ is Davis.*

Proof. Let M be a maximal ideal in A. Then A_M is also Cohen-Macaulay [Ma, Th. 30]. In the local ring A_M every semi-regular sequence is regular [Ma, Th. 31]. Thus the modification $A_M \hookrightarrow A_M[I_M/f] = S^{-1}A'$ (where $S = A \setminus M$) is Davis by Theorem 2.2. Hence $A \hookrightarrow A'$ is Davis by Proposition 2.8. $\square$

Proposition 2.10. *Suppose that $A \hookrightarrow A'$ is a Davis modification. Let A be Cohen-Macaulay. Then A' is also Cohen-Macaulay.*

Proof. Let $L_1, \ldots, L_s \in A^{[s]}$ be as in the proof of Proposition 2.8. Since A is Cohen-Macaulay $A^{[s]}$ is Cohen-Macaulay as well [Ei, Prop. 18.9]. The ideal I' generated by $L_1, \ldots, L_s$ has height s, i.e. its zero set has codimension s in $\operatorname{spec} A^{[s]}$. Hence $A' \simeq A^{[s]}/I'$ is Cohen-Macaulay by [Ei, Prop. 18.13]. $\square$

2.5. **Basic Modifications and Preservation of Normality and UFDs.** We saw (Example 2.1) that semi-basic modifications do not preserve UFDs. Furthermore, they do not preserve normality in general, and we shall need normality.

Example 2.4. Let $A = \mathbf{C}[x, y]$, $f = x^2$ and I is generated by f and y^2. Consider $A \hookrightarrow A'$. Then A' is not normal in the worst possible scenario: X' is given in $\mathbf{C}^3$ by $x^2 z = y^2$ and it has selfintersection points in codimension 1.

We shall show that in the case of Cohen-Macaulay varieties normality survives under basic modifications, but let us emphasize first some nice properties of these modifications.

Remark 2.8. Let $b_0 = f, \ldots, b_s$ be a representative system of generators of a basic modification $A \hookrightarrow A'$. Note $b_0, \ldots, b_s$ may be viewed as elements of a local holomorphic coordinate system at a generic point x of the reduced center of the modification. This implies that every point $y \in \sigma^{-1}(x)$ is a smooth point of X' and the zero multiplicity of $f \circ \sigma$ at y is 1. Actually one can see that locally this modification at x is nothing but a usual (affine) monoidal transformation.

Remark 2.8 and Theorem 2.1 (4) imply

Proposition 2.11. *Let $A \hookrightarrow A'$ be a basic modification. Suppose that C (and, therefore, E) is irreducible and a topological manifold. Suppose also that the natural embedding of C into D generates an isomorphism of the homology of C and D. Then σ generates isomorphisms of the fundamental groups and the homology groups of X and X'.*

Proposition 2.12. *Let $A \hookrightarrow A$ be a basic modification. Suppose that A is normal and Cohen-Macaulay. Then A' is normal and Cohen-Macaulay.*

Proof. By Proposition 2.9 this modification is Davis. Thus by Proposition 2.10 A' is Cohen-Macaulay. Note that if the singularities of X' is at least of codimension 2 then X' is normal by [Ha, Ch. 2, Prop. 8.23]. Since X is normal the codimension of $\sigma^{-1}(\operatorname{sing} X \setminus D) = \sigma^{-1}(\operatorname{sing} X \setminus C)$ in X' is at least 2 whence we can ignore this subvariety. Let C^0 be the subset of the reduced center C, at the points of which the gradients of a representative system of generators are linearly independent. By the definition of a basic modification the codimension of $C \setminus C^0$ in C is at least 1. Since σ is cylindrical the codimension of $\sigma^{-1}(C \setminus C^0)$ in E is at least 1 and in X' is at least 2, and we can ignore these points again. The other points of X' are smooth by Remark 2.8. $\qquad\square$

As soon as we control normality we can take care of preservation of UFDs under affine modifications.

Proposition 2.13. *Let* $A \hookrightarrow A'$ *be an affine modification of normal affine domains such that* E *and* D *are irreducible, and let* $f = g^n$ *where* $g \in A$.

(1) Suppose that A' *is a UFD and the defining ideal of* D *is generated by* g . *Then* A *is a UFD.*

(2) Suppose that A *is a UFD and the defining ideal of* E *in* A' *is generated by the regular function* $g' := g \circ \sigma$ *on* X' . *Then* A' *is a UFD.*

Proof. (1) Let S be a closed irreducible hypersurface in X which is different from D and let S' be its strict transform (i.e. the closure of $\sigma^{-1}(S \setminus D)$ in X'). Since A' is a UFD the defining ideal of S' in A' coincides with the principal ideal generated by a regular function h' on X' . Note that $h' = h/g^k$ where $h \in A$ is not divisible by g . Hence $S \setminus D$ coincides with the zeros of h on $X \setminus D$ and the zeros of h in X does not contain D . The zero multiplicity of h at generic points of S is the same as the zero multiplicity of h' at generic points of S' , i.e. it is 1. If e is another function which vanishes on S then e/h is regular at these generic points and on $X \setminus D$ whence it is regular on X except a subvariety of codimension 2. Hence e/h is holomorphic on X' [Rem, Lemma 13.10] and, therefore, regular (e.g., see [Ka2]). Thus the defining ideal of S is principal and A is a UFD.

(2) Let S' be a closed irreducible algebraic hypersurface in X' . We disregard the case when S' coincides with E since the defining ideal of E is generated by g' . Then $\sigma(S')$ is a constructive set and its closure S is an irreducible hypersurface in X . By the assumption the defining ideal of S in A is the principal ideal generated by a regular function h on X . Suppose that $h' := h \circ \sigma$ has zero multiplicity r at generic points of E . Then $e = h'/(g')^r$ is regular at these generic points (and on $X' \setminus E$, of course). By the same argument about deleting singularities in codimension 2 we conclude that e is regular on X' . By construction its zeros on $X' \setminus E$ coincide with $S' \setminus E$ and these zeros do not contain E . Hence $S' = e^{-1}(0)$ and furthermore the zero multiplicity of e at generic points of S' is 1 (since it is the same of the zero multiplicity of h at generic points of S). Using again the argument about deleting singularities in codimension 2 we see that every regular function which vanishes on S' is divisible by e . Thus X' is a UFD. □

Lemma 2.2. *Let* $A \hookrightarrow A'$ *be an affine modification and* $\dim X = 3$. *Let* $\mathcal{S}'$ *be the germ of an analytic surface in* X' *at* $x' \in E$ *such that* $\mathcal{G}' = \mathcal{S}' \cap E$ *is the germ of*

a curve which meets $\sigma^{-1}(x)$ at x' only, where $x = \sigma(x')$. Then $\sigma(\mathcal{S}')$ is the germ of an analytic surface at $x \in X$.

Proof. By the assumption $x \notin \sigma(\partial \mathcal{S}')$ where $\partial \mathcal{S}'$ is the boundary of $\mathcal{S}'$. Take a small neighborhood V of x in X which does not meet $\sigma(\partial \mathcal{S}')$ and consider the intersection $\mathcal{S}$ of $\sigma(\mathcal{S}')$ and V. Show that it is closed. Consider a sequence of points $x_i \in \mathcal{S}, i = 1, 2, \dots$ which converges to $x_0 \in V$. If this sequence is contained in $\mathcal{G} = \sigma(\mathcal{G}')$ then x_0 belongs to $\mathcal{G}$ since it follows from the assumption that the morphism $\sigma|_{\mathcal{G}'} : \mathcal{G}' \to \mathcal{G}$ is finite. Suppose that none of the points from the sequence is in $\mathcal{G}$ (and, therefore, D). Since $\sigma|_{X' \setminus E} : X' \setminus E \to X \setminus D$ is an isomorphism $x_i' = \sigma^{-1}(x_i)$ is a point in $\mathcal{S}'$. Let $\{x_i'\}$ converges to a point x_0'. Note that $x_0' \notin \partial(\mathcal{S}')$ since otherwise $\sigma(\partial \mathcal{S}')$ meets V. Thus $x_0' \in \mathcal{S}'$ whence $x_0 = \sigma(x_0')$ belongs to $\mathcal{S}$. This implies that $\mathcal{S} \setminus D$ is a closed analytic subset of $V \setminus D$ and none of the irreducible components of the germ of D at x is contained in $\mathcal{S}$. Then by Thullen's theorem (e.g., see [GrRem], th. 2.1) or by Remmert's theorem (e.g., see [BeNa], thm. 1.2) $\mathcal{S}$ is the germ of an analytic hypersurface at $x \in X$. $\qquad \square$

Definition 2.10. We say that X is a local holomorphic UFD if for every $x \in X$ the ring of germs of holomorphic functions at x is a UFD (note that when X is smooth it is a local holomorphic UFD by the theorem of Auslander and Buchsbaum, e.g. see [Ei, Ch. 19, th. 19.19]). Let $\dim X = 3$. We say that X is a local holomorphic UFD with respect to the modification $A \hookrightarrow A'$ if the defining ideal of every germ of analytic surface $\mathcal{S}$ as in Lemma 2.2 is principal in the ring of germs of holomorphic functions. Of course, if X is a local holomorphic UFD then it is a local holomorphic UFD with respect to $A \hookrightarrow A'$.

Proposition 2.14. *Let $A \hookrightarrow A'$ be an affine modification of normal affine domains, and let $f = g^n, g \in A$. Suppose that $A \hookrightarrow A_1$ is another affine modification with locus (J, f) such that $A_1 \subset A'$. Let the defining ideal of the exceptional divisor E_1 in A_1 is generated by $g_1 = g \circ \delta_1$ where $\delta_1 : X_1 \to X$ is the associate morphism of algebraic varieties. Suppose that X is a local holomorphic UFD with respect to $A \hookrightarrow A'$. Then X_1 is a local holomorphic UFD with respect to $A_1 \hookrightarrow A'$.*

The proof of this Proposition is the exact repetition of the proof of Proposition 2.13 (2) with S and S' replaced by $\mathcal{S}$ and $\mathcal{S}_1 = \delta_1^{-1}(S)$.

Remark 2.9. The author believes that if X is a UFD then it is automatically a local holomorphic UFD. This fact is likely known but we could not find a reference, that is why we need Proposition 2.14 besides Proposition 2.13. Furthermore, Proposition 2.14 provides us an additional fact which will be useful later. Let $\mathcal{V} = \mathcal{S} \cap D$ and let

x_1 be the center of the germ $\mathcal{S}_1$, i.e. $\delta_1(x_1) = x$ where x is the center of $\mathcal{S}$. Since X is a local holomorphic UFD with respect to $A \hookrightarrow A'$ there is a small neighborhood V of x in X such that we can suppose that the defining ideal of $\mathcal{S}$ is principal in the ring of holomorphic functions on V. It follows from the proof the defining ideal of $\mathcal{S}_1$ is principal in the ring of holomorphic functions on $V_1 = \delta_1^{-1}(V)$. Note that V_1 is not already a small neighborhood of x_1. It contains the set $\delta_1^{-1}(\mathcal{V})$.

2.6. **Preliminary Decomposition.** We shall fix first notation for this subsection.

Convention 2.2. (1) When we speak about the modification $A \hookrightarrow A'$ in this subsection we suppose that $f = g^n$ where $g \in A$, the zero multiplicity of g at generic points of each irreducible component of D is one, E is non-empty irreducible, and $I = I_f$. Furthermore, we suppose that the defining ideal $\mathcal{I}_{A'}(E)$ of E in A' is generated by g. Recall that by Proposition 2.4 the last condition holds when both A and A' are UFDs and g is irreducible.

(2) Furthermore, we shall consider affine domains $A_i = \mathbf{C}[X_i]$, $i \geq 0$ in this subsection such that $A \hookrightarrow A_i \hookrightarrow A'$. These embeddings generate morphisms of algebraic varieties $\delta_i : X_i \to X$ and $\rho_i : X' \to X_i$ such that $\sigma = \delta_i \circ \rho_i$. By Remark 2.4 there exist ideals I_i in A and K_i in A_i such that $A \hookrightarrow A_i$ is an affine modification with locus (I_i, g^{n_i}) for some $n_i > 0$ and $A_i \hookrightarrow A'$ is an affine modification with locus (K_i, f). Hence the exceptional divisor E_i of the first modification coincides with the divisor D_i of the second one.

(3) We suppose that K_i is the f-largest ideal of the modification $A_i \hookrightarrow A'$ whence by Proposition 2.2 the closure of the geometrical center C_i of ρ_i coincides with its reduced center $\bar{C}_i$.

Lemma 2.3. *Let $A_1 \hookrightarrow A'$ be an affine modification as in Convention 2.2. Suppose that A_1 is normal and the closure of $C_1 = \rho_1(E)$ in X_1 is an irreducible component D_1^1 of D_1. Let E_0 be the Zariski open subset of E that consists of all points $x' \in E$ such that there exists a neighborhood of x' in E which contains no other points from $\rho_1^{-1}(\rho_1(x'))$ but x'. Put $D_0 = \rho_1(E_0)$ and let D_1^2 be the union of irreducible components of D_1 different from D_1^1. Then*

(i) $D_0 = D_1^1 \setminus D_1^2$ and $E_0 = \rho_1^{-1}(D_0)$;

(ii) the restriction of ρ_1 to $(X' \setminus E) \cup E^0$ is an isomorphism between this variety and $(X_1 \setminus D_1) \cup D_0$;

(iii) in particular, if $E = E_0$ (this is so, for instance, when D_1^2 does not meet D_1^1) then ρ_1 is an embedding, and if $D_1 = D_1^1$ then ρ_1 is an isomorphism.

Proof. Since the restriction of ρ_1 to $X' \setminus E$ is an embedding, for every $x' \in E^0$ there exists a Zariski open neighborhood $V_{x'}$ of x' in X' which contains no other points from $\rho_1^{-1}(\rho_1(x'))$ but x'. Put $x_1 = \rho_1(x')$. Since X_1 is normal x_1 cannot be a fundamental point of the birational map ρ_1^{-1} by the Zariski Main Theorem [Ha, Ch. 5, Th. 5.2]. That is, ρ_1^{-1} is a morphism in a neighborhood of x_1 whence ρ_1^{-1} is an embedding in this neighborhood which proves (ii).

We denote by X_1^0 the algebraic subvariety $(X_1 \setminus D_1) \cup (D_1^1 \setminus D_1^2)$. Note that the complement to $(X_1 \setminus D_1) \cup D_0$ in X_1^0 is a constructive subset of codimension at least 2. Since X' is affine and X_1^0 is normal we can extend morphism ρ_1^{-1} to a holomorphic map from X_1^0 to X' [Rem, Lemma 13.10] which is regular (e.g., see [Ka2]). This implies that $\rho_1^{-1}|_{X_1^0} : X_1^0 \to X'$ is an embedding whence $D_0 \supset D_1^1 \setminus D_1^2$. In order to show the reverse inclusion assume that $x' \in E_0$ and $x_1 = \rho_1(x')$ is a point from $D_1^1 \cap D_1^2$. Since ρ_1^{-1} is an embedding in a neighborhood of x_1 we see that the exceptional divisor of ρ_1 must contain a component different from E. This contradiction yields (i). The last statement follows immediately from (i) and (ii). $\square$

We shall need the following technical notion.

Definition 2.11. Let $A \hookrightarrow A'$ be an affine modification, $A \hookrightarrow A_1$ be a basic modification such that $A_1 \subset A'$, and $S = \{h^n \mid n \in \mathbf{N}\}$ be a multiplicative system in A where $h \in A$. Suppose that $(h \circ \sigma)^{-1}(0)$ does not contain E and $S^{-1}A_1 = S^{-1}A'$. Then we call $A \hookrightarrow A'$ a pseudo-basic modification (with respect to $A \hookrightarrow A_1$). That is, this pseudo-basic modification becomes the basic modification $S^{-1}A \hookrightarrow S^{-1}A'$ after the localization.

Note that if the assumption of Lemma 2.3 holds and $A \hookrightarrow A_1$ is basic then it follows from this Lemma that $A \hookrightarrow A'$ is pseudo-basic.

Lemma 2.4. *Let the assumption of Lemma 2.3 and Convention 2.2 hold, and let $A \hookrightarrow A_1$ be a basic modification of rank $s \geq 1$ (i.e. $A \hookrightarrow A'$ is pseudo-basic). Suppose that the reduced center C of σ is a connected component of the reduced center of this modification $A \hookrightarrow A_1$. Then the modification $A \hookrightarrow A'$ is locally basic. Furthermore, if the reduced center of $A \hookrightarrow A_1$ coincides with C then $\rho_1 : X' \to X_1$ is an isomorphism, i.e. $A \hookrightarrow A'$ is basic.*

Proof. Since C is a connected component of the reduced center of $A \hookrightarrow A_1$ and δ_1 is basic (and, therefore, cylindrical by Proposition 2.7) the exceptional divisor $E_1 = D_1$ of δ_1 is of form $D_1^1 \cup D_1^2$ where $D_1^1 = \delta_1^{-1}(C) \simeq \mathbf{C}^s \times C$ and D_1^2 does not meet D_1^1. Lemma 2.3 (iii) implies now the desired conclusion. $\square$

24

Remark 2.10. (1) Note that if C is a point and belongs to the reduced center of δ_1 then C is automatically a connected component of the reduced center of δ_1.

(2) By Proposition 2.12 instead of normality of A_1 one can require that A is normal Cohen-Macaulay.

Proposition 2.15. *Let* $A \hookrightarrow A'$ *and* A_i *be as in Convention 2.2. Suppose also that* C *is not contained in the singularities of* X , *its codimension in* X *is at least 2, and the zero multiplicity of* g *at generic points of* C *is 1. Let*

$$A = A_0 \hookrightarrow \cdots \hookrightarrow A_{k-1} \hookrightarrow A_k, \ k \geq 0$$

is a strictly increasing sequence of affine domains such that $A_k \subset A'$, *and for every* $i \leq k$

(i) *the embedding* $A_{i-1} \hookrightarrow A_i$ *is a basic modification with locus* (J_i, g) *and of rank* s_{i-1} *where* $s_{i-1} + 1$ *is the codimension of* C_{i-1} *in* X_{i-1} *(see Convention 2.2 (3) for the definition of* C_i *).*

(1) *Then* $k \leq n$ *(recall that* $f = g^n$ *) and this sequence can be extended to a strictly increasing sequence of affine domains*

$$A_0 \hookrightarrow \cdots \hookrightarrow A_{m-1} \hookrightarrow A_m, \ k \leq m \leq n$$

for which (i) *holds for every* $i \leq m$, *and* $A_{m-1} \hookrightarrow A'$ *is pseudo-basic with respect to* $A_{m-1} \hookrightarrow A_m$.

(2) *Suppose that* $\sigma_i : X_i \to X_{i-1}$ *is the morphism associated with the affine modification* $A_{i-1} \hookrightarrow A_i$. *Then* $\sigma_i(C_i) = C_{i-1}$ *for* $i \leq m - 1$, *and* $\rho_{m-1}(E) = C_{m-1}$.

(3) *Furthermore, suppose that* A *is normal Cohen-Macaulay,* A' *is normal, and that the closure* E_m^1 *of* $\rho_m(E)$ *is a connected component of* E_m *(resp.* E_m *is irreducible). Then* ρ_{m-1} *is a locally basic (resp. basic) modification.*

Proof. Let us show (2) first. By Convention 2.2 (2) and (3) the exceptional divisor of ρ_i is E whence $\rho_i(E) = C_i$. In particular, $\rho_{m-1}(E) = C_{m-1}$ and $\sigma(E) = C_0$ for $\sigma = \rho_0$. Since $\sigma = \rho_i \circ \delta_i$ we see that $\delta_i(C_i) = C_0$. This implies that $\sigma_i(C_i) = C_{i-1}$ since $\delta_i = \delta_{i-1} \circ \sigma_i$.

If $s_k = 0$ then we put $m = k$ and get (1) automatically. Otherwise, let us show now that the assumptions of this Proposition hold if we replace $A \hookrightarrow A'$ by $A_k \hookrightarrow A'$. It is enough to check this for $k = 1$. By Remark 2.8 for a generic point $x \in \rho_0(E) = \sigma(E)$ the points from $\sigma_1^{-1}(x)$ are smooth in X_1 . Since $\rho_0 = \sigma_1 \circ \rho_1$ we see that $\sigma_1^{-1}(x)$ contains generic points of $\rho_1(E)$. Hence generic points of C_1 are not contained in the singularities of X_1 , and the zero multiplicity of the function $g \circ \delta_1$ at these generic points is 1 by Remark 2.8. By Lemma 2.1 and Proposition 2.3 we can choose a basic

modification $A_k \hookrightarrow A_{k+1}$ with locus (J_k, g) such that $A_{k+1} \subset A'$ and the rank of the modification is s_k. Thus we can extend our strictly increasing sequence of affine domains and we can always suppose that $k \geq 1$ in (1). There are two possibilities: either this sequence becomes eventually infinite or there exists m such that $s_m = 0$ which implies (1). We need to show that the first possibility does not hold and that $m \leq n$ with help of induction by n.

Assume first that (1) holds for $n - 1 > 0$ and show it for n. Let $b_0 = g, b_1, \ldots, b_s$ be a representative system of generators for $A \hookrightarrow A_1$. By assumption of this Proposition and Definition 2.5 there exists $h \in A$ such that $h^{-1}(0)$ does not contain $\sigma(E)$, $X \setminus h^{-1}(0)$ is smooth, $C \setminus h^{-1}(0)$ is a complete intersection in $X \setminus h^{-1}(0)$ given by $b_0 = \cdots = b_s = 0$. If S is the multiplicative system $\{h^j \mid j \in \mathbf{N}\}$ in A then the affine modification $S^{-1}A \hookrightarrow S^{-1}A'$ satisfies the analogue of assumption of this Proposition and, furthermore, $S^{-1}J_1$ is the defining ideal of $C \setminus h^{-1}(0)$ in $S^{-1}A$. Since the locus of the affine modification $S^{-1}A \hookrightarrow S^{-1}A_1$ is $(S^{-1}J_1, g)$ by Proposition 2.1, Corollary 2.1 implies that for the affine modification $S^{-1}A_1 \hookrightarrow S^{-1}A'$ the locus can be chosen in the form (L_1, g^{n-1}). Thus after the localizations of our strictly increasing sequence of affine domains with respect to S we have by the induction assumption that the codimension of the reduced center of the modification $S^{-1}A_m \hookrightarrow S^{-1}A'$ is 1 for some $m \leq n$. This implies that the same is true for the reduced center of $A_m \hookrightarrow A'$, i.e. $s_m = 0$ which concludes this step of induction.

The next step of induction is for $n = 1$. By Proposition 2.3 in this case $S^{-1}J_1$ coincides with the g-largest ideal of the affine modification $S^{-1}A \hookrightarrow S^{-1}A'$. Hence $S^{-1}A_1 = S^{-1}A'$. Since h is chosen so that $h^{-1}(0)$ does not contain C this implies (1) which concludes induction.

Note that when A is normal Cohen-Macaulay so is A_k by Propositions 2.9, 2.10, and 2.12. Claim (3) is now a consequence of Lemma 2.4. $\square$

Let C_{m-1}^* be the complement in C_{m-1} to the set of points where C_{m-1} meets the other components of the reduced center of σ_m. Then the exceptional divisor of the basic modification σ_m contains $E_m^* \simeq \mathbf{C}^{s_{m-1}} \times C_{m-1}^*$. Furthermore, under the assumption of Proposition 2.15 (3) the restriction of ρ_m^{-1} to $(X_m \setminus E_m) \cup E_m^*$ is an embedding by Lemma 2.3. Hence

Corollary 2.2. (cf. [Miy2, Lemma 2.3]) *Under the assumption of Proposition 2.15 (3) the exceptional divisor E contains a Zariski open cylinder $E_m^* \simeq \mathbf{C}^{s_{m-1}} \times C_{m-1}^*$ such that $\rho_{m-1}|_{E_m^*}$ is the projection to the second factor.*

2.7. **Decomposition.** In this subsection we shall strengthen Proposition 2.15 in the case when $\dim X = 3$ and X is a holomorphic UFD. Our main aim is to make $A_m = A'$ in this Proposition. By Lemma 2.4 it is enough to require that the reduced center of σ_m is irreducible, i.e. it coincides with $\bar{C}_{m-1}$. This is true when the defining ideal of $\bar{C}_{m-1}$ in $\mathbf{C}[D_{m-1}]$ is principal provided C_{m-1} is a curve, or more generally the defining ideal of each C_i in $\mathbf{C}[D_i]$ is principal provided C_i is a curve. The last claim will be proven by induction and the first step of induction is crucial. But the proof of this step is difficult and we present it in the next section (Proposition 3.1). Another non-trivial fact whose proof is postponed till next section says that the number of irreducible components of the germ of C_{m-1} at each point $z \in C_{m-1}$ coincides with the number of connected components in $\rho_{m-1}^{-1}(z)$ (Lemma 3.3). This helps us to show that C_i's are contractible in some cases. Furthermore, if we want to check that C_i's are smooth we need the following

Lemma 2.5. *Let Convention 2.2 hold and let $A_1 \hookrightarrow A'$ be a basic modification with the divisor D_1 of modification isomorphic to $\mathbf{C} \times C$ where C is a curve. Suppose that C has an irreducible singular point z and that C_1 meets the line $\mathbf{C} \times z$ in D_1 at $z_1 = 0 \times z$ but C_1 is different from this line. Then z_1 is a singular point of C_1.*

Proof. Assume the contrary, i.e. C_1 is smooth at z_1. Since the situation is local we can suppose that C is a closed curve in $\mathbf{C}^n$. Consider a normalization $\nu_0 : C^\nu \to C$. It generates a morphism $\nu = (\mathrm{id}, \nu_0) : \mathbf{C} \times C^\nu \to \mathbf{C} \times C \subset \mathbf{C}^{n+1}$. Suppose that $(y, \bar{x}) = (y, x_1, \ldots, x_n)$ is a coordinate system in $\mathbf{C}^{n+1}$. Let g, b_1 be an almost complete intersection in A_1 which generates this basic modification $A_1 \hookrightarrow A'$, i.e. b_1 generates the defining ideal of C_1 in $\mathbf{C}[D_1]$. We treat b_1 as a polynomial $b_1(y, \bar{x})$ on $\mathbf{C}^{n+1}$. Let $\beta = b_1|_{D_1} \circ \nu$, C_1^ν be the proper transform of C_1 (i.e. $C_1^\nu = \beta^{-1}(0)$), and let $o = \nu^{-1}(z_1)$. Since C_1 is smooth (and, therefore, normal) and since $\nu|_{C_1^\nu} : C_1^\nu \to C_1$ is a homeomorphism (i.e., this morphism is proper and finite), C_1^ν is biholomorphic to C_1 (e.g., see [Pe1, Cor. 1.5]) whence C_1^ν is smooth at o. Since the modification is basic the condition on the gradients implies that the gradient of β does not vanish at generic points of C_1^ν. Hence since C_1^ν is smooth at o the gradient of β does not vanish at o. Let (v, t) be a local coordinate system at o where t is a coordinate on the second factor and v is a coordinate on the first factor of $\mathbf{C} \times C^\nu$. In particular, locally $\nu(v, t) = (v, \bar{x}(t))$ whence the Taylor series of $\beta(v, t) = b_1(v, \bar{x}(t))$ at o does not have a nonzero linear term ct with $c \in \mathbf{C}$ (recall that z is a singular point of C whence none of $x_i(t)$'s contains a linear term ct). Hence, the linear part of this power series must be v up to a nonzero constant factor (the factor is nonzero since otherwise the gradient of β at o is zero). Thus the

Taylor series of b_1 at z_1 has a nonzero linear term cv. The implicit function theorem implies that the germ of C_1 at z_1 is isomorphic to the germ of C at z whence C_1 is singular at z_1. Contradiction. $\qquad\square$

Theorem 2.3. *Let $A \hookrightarrow A'$ be an affine modification such that Convention 2.2 (1) holds (recall that this is true when E is non-empty, A and A' are UFDs and $f = g^n$ where $g \in A$ is irreducible), A is Cohen-Macaulay, and $\dim X = 3$. Suppose that either*

(α) X' is smooth and $H_3(X') = H_2(X' \setminus E) = H_3(X' \setminus E) = 0$, or

(β) E is a UFD and its Euler characteristics is 1.

Suppose also that

(i) D is isomorphic to $\mathbf{C}^2$; [9]

(ii) A is a local holomorphic UFD (see Definition 2.10) with respect to the modification $A \hookrightarrow A'$ (which is true when X is smooth);

(iii) C is not contained in the singularities of X.

Let m, A_i, C_i, and $\bar{C}_i$ be the same as in Proposition 2.15 and Convention 2.2 (3).

(1) Then the algebras A_i 's can be chosen so that $A_m = A'$, $C_i = \bar{C}_i$ for every i, and if C_i is a curve its defining ideal in $\mathbf{C}[D_i]$ is principal.

(2) Furthermore, each C_i is either a point or an irreducible contractible curve, and in the case when E has at most isolated singularities (which is true under condition (β)) these contractible curves are smooth.

Proof. We use induction by m. Suppose first that C_0 is a point. The assumption (i) on D and the fact that C_0 is a smooth point of X by (iii) allow us to choose $b_1, b_2 \in A$ such that g, b_1, b_2 generate the defining ideal I_1 of C_0 in A. Hence the exceptional divisor $E_1 = D_1$ of the basic modification $A \hookrightarrow A_1 = A[I_1/g]$ with locus (I_1, g) is isomorphic to $\mathbf{C}^2$ and, in particular, irreducible. Note that $A \subset A'$ by Proposition 2.3 and Convention 2.2 (1). If $m = 1$ in Proposition 2.15 then Lemma 2.3 implies that ρ_1 is an isomorphism, and, in particular, $A_1 = A'$.

Suppose now that $m \geq 2$. Note that E_1 is isomorphic to $\mathbf{C}^2$ and it is the divisor of the modification $A_1 \hookrightarrow A'$ described in Convention 2.2. By Propositions 2.10 and 2.12 A_1 is normal Cohen-Macaulay. By Proposition 2.13 A_1 is a UFD and furthermore it is a local holomorphic UFD with respect to the modification $A_1 \hookrightarrow A'$ by Proposition 2.14. Thus the assumptions of this Theorem hold also for the modification $A_1 \hookrightarrow A'$.

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[9]One can replace (i) with the condition that D is a UFD. In this case the statement of the theorem remains the same with one exception : when C_0 is a point and we want the center of σ_1 to be the defining ideal of C_0 in A then we have to allow σ_1 to be not necessarily basic but only locally basic.

The decomposition of this last modification into basic modifications contains $m - 1$ factors and induction implies the desired conclusion in this case.

If $\bar{C}_0$ is a curve then its defining ideal in $\mathbf{C}[D]$ is principal whence the defining ideal I_1 of $\bar{C}_0$ in A is generated by g and $b \in A$. The exceptional divisor E_1 of $A \hookrightarrow A_1 = A[I_1/g] \subset A'$ is again irreducible, and $A_1 \subset A'$ as before. If $m = 1$ then Lemma 2.3 implies that ρ_1 is an isomorphism, i.e. $A_1 = A'$. Furthermore, for every $z \in C_0$ the number of components in $\sigma^{-1}(z)$ is one, since $\sigma^{-1}(z) \simeq \sigma_1^{-1}(z) \simeq \mathbf{C}$ (recall that σ_1 is basic). Hence Lemma 3.3 below implies that the number of irreducible components of the germ of C_0 at z is one, i.e. z is not a double point of C_0. The same Lemma says that the normalization of C_0 is $\mathbf{C}$ whence C_0 is contractible. This means that C_0 is closed in the ambient affine algebraic variety D, i.e. $C_0 = \bar{C}_0 = C$. When C_0 has singularities then E has singularities in codimension 1 which yields the last statement.

Actually in the case when C_0 is a curve, instead of (i) one can assume that D is isomorphic to $\mathbf{C} \times G$ where G is a curve and the natural projection $C \to G$ is dominant (this is, of course, true for $D \simeq \mathbf{C}^2$). This curve C_0 is contractible by Proposition 3.1 below (which implies, in particular, that $C = C_0$) whence the projection $C \to G$ is finite. The defining ideal of C_0 in $\mathbf{C}[D]$ is generated by a function $b \in A$ (see Proposition 3.1 below) whence the defining ideal I_1 of C in A is generated by g and b. Thus the exceptional divisor E_1 of the basic modification $A \hookrightarrow A_1 = A[I_1/g]$ is again irreducible and we see as before that A_1 is a Cohen-Macaulay UFD which is also a local holomorphic UFD with respect to the modification $A_1 \hookrightarrow A'$. Thus when $m \geq 2$ the assumption of this Theorem holds for $A_1 \hookrightarrow A'$ and induction implies statement (1).

The curves C_i's are contractible for $i \geq 2$ by the induction assumption. Lemma 2.5 implies that if C_0 is not smooth then C_1 cannot be smooth which yields the last statement of (2). $\qquad\square$

3. The Geometry of The Exceptional Divisor and The Reduced Center

3.1. The Exceptional Divisor.
We shall finish the proof of Theorem 2.3 in this section. First we describe E in the three-dimensional case more accurately then Corollary 2.2 does.

Lemma 3.1. *Let $A \hookrightarrow A'$ be an affine modification such that E is irreducible and the geometrical center C_0 of this modification is a curve. Suppose that X, X' are of dimension 3. Let $A \hookrightarrow A_1$ be a semi-basic modification of rank 1 and with*

locus (I_1, f) *such that* $A_1 \subset A'$. *Then for every finite subset* $R \subset C_0$ *there exists a semi-basic modification* $A \hookrightarrow A_2$ *of rank 1 with locus* (I_2, f) *such that* $A_2 \subset A'$, *and* ρ_2 *is not constant on each of irreducible components of* $\sigma^{-1}(R)$ *where* ρ_2 *is as Convention 2.2 (2).*

Proof. Let a semi-regular sequence $f, b_1 \in A$ generate I_1, i.e. $I_1 \subset I$ and $A_1 = A[I_1/f]$. Consider several basic modifications of this type. That is, for $j = 1, \ldots, k$ the sequence $b_0 = f, b_j$ is semi-regular and it generates an ideal $I_j \subset I$. Let $A \hookrightarrow A_j = A[I_j/f]$ be an affine modification with locus (I_j, f). Recall that $\delta_j : X_j \to X$ is the corresponding morphism of affine algebraic varieties and E_j is the exceptional divisor of δ_j. These morphisms $\{\delta_j\}$ define an affine variety $Y = X_1 \times_X X_2 \times_X \cdots \times_X X_k$ and its subvariety $Y^* = (X_1 \setminus E_1) \times_X \cdots \times_X (X_k \setminus E_k)$. Since we can perturb the elements b_j of our semi-regular sequences by Lemma 2.1 (4), we can suppose that $I_1 + \cdots + I_k = I$. By Proposition 2.6 X' can be viewed as the closure $\bar{Y}^*$ of Y^* in Y, σ can be viewed as the restriction of the natural projection $Y \to X$ to $\bar{Y}^*$, and ρ_j (from Convention 2.2 (2)) will be nothing but the restriction of the natural projection $Y \to X_j$ to $\bar{Y}^*$.

Note that X_j can be viewed as a closed subvariety of $\mathbf{C} \times X$ such that a coordinate y_j on the first factor of $\mathbf{C} \times X$ is chosen so that its restrictions to X_j coincides with b_j/f, and δ_j coincides with the restriction of the natural projection $\mathbf{C} \times X \to X$. Thus X' can be viewed as a closed subvariety of $\mathbf{C}^k \times X$ such that $(y_1, \ldots, y_k)$ is a coordinate system of the first factor, the restriction of y_j to X' is b_j/f, and σ coincides with the restriction of the natural projection $\mathbf{C}^k \times X \to X$ to $X' (= \bar{Y}^*)$.

Let $x \in C_0$. Since $\dim \sigma^{-1}(x_1) = 1$ for a generic point $x_1 \in C_0$ the dimension of every irreducible component of $\sigma^{-1}(x)$ is at least 1 by the semi-continuity of the dimension of the fibers of an algebraic morphism. But this dimension cannot be equal to 2 since otherwise E contains at least two irreducible components in contradiction with the assumption of this Proposition. Thus $\sigma^{-1}(x)$ is a curve. Since $\sigma(\sigma^{-1}(x)) = x$ the image of $\sigma^{-1}(x)$ in $\mathbf{C}^k$ under the natural projection is a curve, i.e. the natural projection of $\sigma^{-1}(x)$ to a generic affine line in $\mathbf{C}^k$ is dominant. By Lemma 2.1 after a small perturbation of b_2 in I the sequence b_0, b_2 remains a semi-regular sequence, i.e. we can suppose that the y_2-axis is a generic line. Thus if $\tau : X' \to \mathbf{C}$ is the natural projection to the y_2-axis then we can suppose that τ is not constant on every irreducible component of $\sigma^{-1}(R)$. Note that $\tau = \theta_2 \circ \rho_2$ where $\theta_2 : X_2 \to \mathbf{C}$ is the the natural projection to the y_2-axis. Hence the restriction of ρ_2 to every component of $\sigma^{-1}(R)$ is not constant. $\qquad \square$

Lemma 3.2. *Let Convention 2.2 hold (in particular, C_0 is the geometrical center of σ) and X_1 and X' be normal varieties of dimension 3. Let $A \hookrightarrow A'$ be a pseudo-basic modification with respect to a basic modification $A \hookrightarrow A_1$.*

(1) Then for every $z \in C_0$ the curve $\sigma^{-1}(z)$ is a disjoint union of irreducible contractible curves.

(2) If E has no double points then it is a topological manifold.

(3) If in addition to (2) for every $z \in C_0$ the curve $\sigma^{-1}(z)$ is connected then E is naturally homeomorphic to the product of $\mathbf{C}$ and a curve.

Proof. By Corollary 2.2 there exists $C^* \subset C$ for which E contains a Zariski open cylinder $E^* \simeq \mathbf{C} \times C^*$ such that $\sigma|_{E^*}$ is the projection to the second factor. Therefore, we need to consider only z from the finite set $R = C_0 \setminus C^*$. Let (g, d) be an almost complete intersection which generates the basic modification $A \hookrightarrow A_1$. Then $f = g^n, (b_1)^n$ is a semi-regular sequence which is contained in I . Hence we can choose a semi-regular sequence f, b_2 in I which generates a semi-basic modification $A \hookrightarrow A_2$ as in Lemma 3.1.

Consider $Y = X_1 \times_X X_2$ and and its subvariety $Y^* = (X_1 \setminus E_1) \times_X (X_2 \setminus E_2)$. Let X_0 be the closure $\bar{Y}^*$ of Y^* in Y . Recall that X_1 (resp. X_2) can be viewed as a closed subvariety of $\mathbf{C} \times X$ such that a coordinate y_1 (resp. y_2) on the first factor of $\mathbf{C} \times X$ is chosen so that its restrictions to X_1 (resp. X_2) coincides with d/g (resp. b_2/f), and δ_j coincides with the restriction of the natural projection $\mathbf{C} \times X \to X$. Thus X_0 can be viewed as a closed subvariety of $\mathbf{C}^2 \times X$ such that (y_1, y_2) is a coordinate system of the first factor.

Let $\delta_0 : X_0 \to X$, $\tau_i : X_0 \to X_i$ be the natural projections (note that δ_0 coincides with the restriction of the natural projection $\mathbf{C}^2 \times X \to X$ to X_0). Put $E_i^* = \delta_i^{-1}(C^*)$. Let F be the closure of E_0^* in $\mathbf{C}^2 \times X$. Note that $\delta_0 = \delta_i \circ \tau_i$ and $\sigma = \delta_i \circ \rho_i$. Hence $\rho_1 = \tau_1 \circ \rho_0$. By Corollary 2.2 $\rho_1|_{E^*} : E^* \to E_1^*$ is an isomorphism whence $\rho_0|_{E^*} : E^* \to E_0^*$ and $\tau_1|_{E_0^*} : E_0^* \to E_1^*$ are isomorphisms.

Let us show that $\rho_0(\sigma^{-1}(z))$ is contained in a disjoint union of contractible lines. Note that $F \subset X_0$ and $\rho_0(E) \subset F$. Hence $\rho_0(\sigma^{-1}(z))$ is contained in $F \setminus E_0^*$. Since δ_1 is basic $E_1^* \simeq \mathbf{C} \times C^*$ where y_1 is a coordinate on the first factor. The surface F is contained in $\mathbf{C}^2 \times C$ where (y_1, y_2) is a coordinate system on the first factor. Since $\tau_1|_{E_0^*} : E_0^* \to E_1^*$ is an isomorphism and coincides with the restriction of the natural projection $\mathbf{C}^2 \times C \to \mathbf{C} \times C, ((y_1, y_2), x) \to (y_1, x)$ for every $x \in C^*$ the equation of $\tau_1^{-1}(x)$ in $\mathbf{C}^2_{y_1, y_2}$ is of form $\bar{a}(x)y_2 + \sum_{i=0}^k \bar{a}_i(x)y_1^i = 0$ where $\bar{a}, \bar{a}_i$ are regular functions on C^* and $\bar{a}$ is invertible. Consider $z \in C \setminus C^*$ and an irreducible branch $\mathcal{C}$ of of the germ of C at z . Let $\bar{a}'$ be one of the restrictions of $\bar{a}_i$'s or

$\bar{a}$ to $\mathcal{C} \setminus z$ which has the largest pole at z. Dividing by $\bar{a}'$ we see that the curves $\bar{a}(x)y_2 + \sum_{i=0}^{k} \bar{a}_i(x)y_1^i = 0, x \in \mathcal{C}$ approach to a curve

$$a(z)y_2 + \sum_{i=0}^{k} a_i(z)(y_1)^i = 0 \tag{1}$$

where $a(x)$ and $a_i(x)$'s are rational continuous functions on $\mathcal{C}$ which are regular on $\mathcal{C} \setminus z$. Not all coefficients before y_i's in equation (1) are zeros and we have three cases.

Case 1. If $a_0(z) \neq 0$ and the rest of coefficients are zeros then equation (1) defines an empty set.

Case 2. If $a(z) = 0$ and some $a_i(z) \neq 0$ for $i \geq 1$ then equation (1) defines a set of line parallel to the y_2-axis.

Case 3. If $a(z) \neq 0$ then equation (1) defines a contractible irreducible curve T such that y_1 is a coordinate on T. If T is the closure of the image of an irreducible component of $\sigma^{-1}(z)$ under ρ_0 then the restriction of ρ_1 to this component gives a dominant morphism to $\delta_1^{-1}(z)$ (this last curve is isomorphic to $\mathbf{C}$ since δ_1 is basic). By Lemma 2.3 (and in its notation) $\delta_1^{-1}(z)$ is contained in D_0 whence ρ_1^{-1} is an embedding in a neighborhood of $\delta_1^{-1}(z)$. Since $\rho_1 = \rho_0 \circ \tau_1$ we see that ρ_0^{-1} is an embedding in a neighborhood of $\delta_0^{-1}(z)$ and τ_1^{-1} is an embedding in a neighborhood of $\delta_1^{-1}(z)$. That is, in this case z can be treated as a point of C^*. Thus $\rho_0(\sigma^{-1}(z))$ is contained in a disjoint union of contractible lines.

Note that every irreducible component T' of $\sigma^{-1}(z)$ is a limit of curves $\sigma^{-1}(x), x \in C^*$ which are isomorphic to $\mathbf{C}$. Thus T' admits a non-constant morphism from $\mathbf{C}$, i.e. T' is a once punctured curve. This implies that ρ_0 maps T' surjectively on an irreducible component T of $F \setminus E_0^*$ (recall that ρ_0 is non-constant by Lemma 3.1). Hence it remains to show that $\rho_0|_E : E \to F$ is an injection. Assume the contrary, i.e. there exist different points x_1', x_2' in $E \setminus E^*$ such that $\rho_0(x_1') = \rho_0(x_2') = x \in T$. Let V_i be a neighborhood of x_i' in E. Since ρ_0 is non-constant on every component of $\sigma^{-1}(z)$ we see that $\rho_0(V_i)$ contains the germ $\mathcal{T}$ of T at x and for a generic point $x_0 \in \mathcal{T}$ a neighborhood of x_0 is contained in $\rho_0(V_i)$. Thus $(\rho_0(V_1) \cap \rho_0(V_2)) \setminus T$ is not empty. Since $\rho_0|_{E^*} : E^* \to E_0^*$ is an isomorphism we see that V_1 and V_2 meet each other. Hence E is not separable which is not true since E is affine. Thus $E^* \to F$ is an injection which implies (1).

Note that if E has no double points then, since $E^* \to F$ is an injection, the equation $a(x)y_2 + q(x, y_1) = 0$ with $q(x, y_1) = \sum_{i=0}^{k} a_i(x)(y_1)^i$ and x running over $\mathcal{C}$ defines a homeomorphic image of a germ of E. Thus it suffices to check the

statements (2) and (3) for the variety given this equation in Case 2 (two other cases are obvious).

Replacing $\mathcal{C}$ by its normalization one can suppose that $\mathcal{C}$ is smooth and, therefore, may be viewed as the germ of $\mathbf{C}$ at 0. Thus x can be treated as a coordinate on $\mathbf{C}$ now. If k is the multiplicity of zero of $a(x)$ we can replace this equation $a(x)y_2 + q(x, y_1) = 0$ with $x^k y_2 + q(x, y_1) = 0$. The last equation defines a variety Z. Put $q_1(x, y_1) = \frac{\partial}{\partial y_1} q(x, y_1)$. Let $\{c_i\}$ be the roots of $q(0, y_1)$ and let l be the zero multiplicity of $q_1(x, c_1)$. Let Z_1 be the variety obtained from Z by deleting all lines $x = y - c_i = 0$ where $i \geq 2$. Consider the following locally nilpotent derivation on the algebra of regular functions on Z_1: $\partial(x) = 0, \partial(y_1) = -x^{k-l}, \partial(y_2) = q_1(x, y_1)/x^l$. It defines a regular $\mathbf{C}_+$-action on Z_1 which maps the line $x = y - c_1 = 0$ onto itself and acts transitively on this line. Thus the preimage of this line in a normalization Z_1^ν of Z_1 does not contain singular points of Z_1^ν. Indeed, otherwise each point of this preimage is singular (by the transitivity of the $\mathbf{C}_+$-action) and the normal variety Z_1^ν has singularities in codimension 1 which cannot be true. Hence Z_1^ν is smooth. Since in the absence of double points any normalization is a homeomorphism Z_1 is a topological manifold which yields (2). Furthermore, it is easy to see that Z_1^ν is isomorphic to $\mathcal{C} \times \mathbf{C}$ which implies (3). $\square$

Lemma 3.3. *Let X' be an affine threefold with $H_3(X') = 0$ and E be a closed irreducible surface in X' which admits a surjective morphism $\tau : E \to C_{m-1}$ into a curve C_{m-1}* [10] *such that for a Zariski open subset $C_{m-1}^* \subset C_{m-1}$ and $E^* = \tau^{-1}(C_{m-1}^*)$ the morphism $\tau|_{E^*} : E^* \to C_{m-1}^*$ is a $\mathbf{C}$-cylinder and $L := E \setminus E^*$ is a disjoint union of irreducible contractible curves. Let $H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. Suppose that $z \in C_{m-1} \setminus C_{m-1}^*$ and $\mathcal{C}_z$ is the germ of C_{m-1} at z (we treat $\mathcal{C}_z$ as a bouquet of discs). Put $\mathcal{C}^z = \mathcal{C}_z \setminus z$, $\mathcal{E}^z = \tau^{-1}(\mathcal{C}^z)$, $\mathcal{E}_z = \tau^{-1}(\mathcal{C}_z)$, and $L^z = L \cap \mathcal{E}_z \,(= \tau^{-1}(z))$. Then there exists an isomorphism $H_0(L) \simeq H_1(E^*)$ such that for every germ $\mathcal{C}_z$ as above the restriction of this isomorphism generates an isomorphism $H_0(L^z) \simeq H_1(\mathcal{C}^z)$ (in particular, the number of connected components of L^z is the same as the number of irreducible components of $\mathcal{C}_z$). Furthermore, the normalization of C_{m-1} is $\mathbf{C}$.*

Proof. Let $L := E \setminus E^*$. Consider the following exact homology sequences of pairs:

$$\longrightarrow H_{j+1}(X') \to H_{j+1}(X', X' \setminus L) \to H_j(X' \setminus L) \to H_j(X') \to H_j(X', X' \setminus L) \to$$

and

$$\ldots \longrightarrow H_j(X' \setminus E) \to H_j(X' \setminus L) \to H_j(X' \setminus L, X' \setminus E) \to H_{j-1}(X' \setminus E) \to .$$

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[10] We denote this curve by C_{m-1} since it will play later the role of the geometrical center of the modification ρ_{m-1} from Proposition 2.15 and τ will be $\rho_{m-1}|_E$.

Note that $H_4(X') = 0$ since X' is an affine algebraic variety [Mil, th. 7.1]. Taking into consideration the other assumptions on $H_j(X' \setminus E)$ and $H_j(X')$ and Thom's isomorphisms $H_0(L) \simeq H_4(X', X' \setminus L)$ and $H_1(E^*) \simeq H_3(X' \setminus L, X' \setminus E)$ (e.g. see [Do, Ch. 8, 11.21]) we have

$$H_0(L) \simeq H_4(X', X' \setminus L) \simeq H_3(X' \setminus L) \simeq H_3(X' \setminus L, X' \setminus E) \simeq H_1(E^*).$$

Note also that $H_1(C^*_{m-1}) \simeq H_1(E^*)$ whence we have an isomorphism between $H_0(L)$ and $H_1(C^*_{m-1})$.

Let us have a closer look at this isomorphism. Suppose that L_i is an irreducible component of L . Consider the germ $\mathcal{S}'_i$ of a smooth complex surface which is transversal to both E and L_i at a smooth z' point of L_i . Recall that the Thom class is the element $u_i \in H^4(X', X' \setminus L_i)$ uniquely defined by the condition $u_i(\mathcal{S}'_i) = 1$. The Thom isomorphism $H_4(X', X' \setminus L) \to H_0(L)$ is defined by the cap-product with the Thom class $u = \sum_i u_i : \eta \to u \cap \eta$ (in particular, $\mathcal{S}'_i$ as an element of $H_4(X', X' \setminus L)$ is mapped under this isomorphism to the positive generator of $H_0(L_i)$). We can suppose that $\mathcal{S}'_i$ is diffeomorphic to a ball and its boundary $\partial \mathcal{S}'_i$ in X' is diffeomorphic to a three-sphere. The isomorphism $H_4(X', X' \setminus L) \simeq H_3(X' \setminus L)$ sends $\mathcal{S}'_i$ to $\partial \mathcal{S}'_i$ (which is viewed as an element of $H_3(X' \setminus L)$). Then we can suppose that $\mathcal{S}'_i$ meets E^* transversally along a disjoint union of the germs of curves $\Gamma_i = \bigcup_j \Gamma_i^j$, that the boundary of each germ Γ_i^j in E^* is a smooth closed real curve γ_i^j , and that $\partial \mathcal{S}'_i$ meets E^* transversally along $\gamma_i = \bigcup_j \gamma_i^j$. Since $\mathcal{S}'_i$ is a germ and $\tau(z') = z$ we see that $\gamma_i \subset \mathcal{E}^z$. Let T_i be a small neighborhood of γ_i in $\partial \mathcal{S}'_i$. By the excision theorem T_i generates the same element of $H_3(X' \setminus L, X' \setminus E)$ as $\partial \mathcal{S}'_i$ does. Thus the isomorphism $H_0(L) \simeq H_3(X' \setminus L, X' \setminus E)$ sends the generator of $H_0(L_i)$ to T_i . The advantage of T_i is that it can be chosen as a fibration over γ_i where for each j the fiber of T_i over the curve γ_i^j can be viewed as a small complex disc Δ_i^j which meets E^* transversally at a point of γ_i^j . Furthermore, the connected component of T_i over γ_i^j is naturally diffeomorphic to $\Delta_i^j \times \gamma_i^j$. Recall that the Thom class v_i^j of γ_i^j in $X' \setminus L$ is an element of $H_2(X' \setminus L, X' \setminus E)$ uniquely defined by the condition $v_i^j(\Delta_i^j) = 1$. The Thom isomorphism $H_3(X' \setminus L, X' \setminus E) \to H_1(E^*)$ is defined by the cap-product with the Thom class $v = \sum_{i,j} v_i^j : \theta \to v \cap \theta$. The properties of the cap-product [Do, Ch. 7, 12.5 and 12.6] imply that the image of T_i is γ_i which is viewed as an element of $H_1(E^*) \simeq H_1(C^*_{m-1})$. Note that $H_1(C^*_{m-1}) = \oplus_{z \in C_{m-1} \setminus C^*_{m-1}} H_1(\mathcal{C}^z) \oplus N$ where the group N is not trivial provided that either C_{m-1} is of positive genus or C_{m-1} has more than one puncture. Since $\gamma_i \subset \mathcal{E}^z$ we see that the image of the generator of $H_0(L_i)$ under the isomorphism $H_0(L) \simeq H_1(C^*_{m-1})$ is contained in $H_1(\mathcal{C}^z)$.

34

Hence the image of $H_0(L)$ is contained in $\oplus_{z \in C_{m-1} \setminus C^*_{m-1}} H_1(\mathcal{C}^z)$. Thus N is trivial and $H_0(L^z) \simeq H_1(\mathcal{C}^z)$. This is the desired conclusion. $\qquad \square$

The proof of Lemma 3.3 implies more. Let $\mathcal{C}_j$, $j = 1, \ldots, k$ be the irreducible components of $\mathcal{C}_z$. Then $\mathcal{C}_j$ corresponds to a generator α_j of $H_1(\mathcal{C}^z)$. If L_i is an irreducible component of $\tau^{-1}(z)$ then it corresponds to a generator β_i of $H_0(L^z)$. By Lemma 3.3 the image of β_i under the isomorphism $H_0(L^z) \simeq H_1(\mathcal{C}^z)$ is $\sum_j m_i^j \alpha_j$. One can extract from Lemma 3.3 the way to compute these coefficients m_i^j.

Lemma 3.4. *Let the notation above hold and let $\mathcal{S}_i'$ be a germ of a holomorphic smooth surface which is diffeomorphic to a complex two-dimensional ball and transversal to L_i and E' at a generic point x' of L_i. Suppose $\mathcal{E}_j$ is the closure of $\tau^{-1}(\mathcal{C}_j \setminus z)$ and $\mathcal{S}_i'$ meets $\mathcal{E}_j$ along a curve Γ_i^j. Then the mapping $\tau|_{\Gamma_i^j} : \Gamma_i^j \to \mathcal{C}_j$ is m_i^j-sheeted where m_i^j is as before this Lemma.*

Proof. One can suppose that the boundary $\partial \mathcal{S}_i'$ of $\mathcal{S}_i'$ meets $\mathcal{E}_j$ transversally along a closed real curve γ_i^j. It was shown in the proof of Lemma 3.3 that the image of β_i under the isomorphism $H_0(L^z) \simeq H_1(\mathcal{E}^z)$ is $\sum_j \gamma_i^j$ where γ_i^j is viewed as an element of $H_1(\mathcal{E}^z)$. Then the image of γ_i^j under the isomorphism $H_1(\mathcal{E}^z) \simeq H_1(\mathcal{C}^z)$ coincides with $m_i^j \alpha_j$ where m_i^j is the winding number of $\tau(\gamma_i^j)$ in $\mathcal{C}_j$ around z. On the other hand γ_i^j is the boundary of Γ_i^j. This implies that $\tau|_{\Gamma_i^j} : \Gamma_i^j \to \mathcal{C}_j$ is m_i^j-sheeted. $\qquad \square$

Remark 3.1. (1) If E is a UFD then there is no need to assume in Lemma 3.3 that X' is smooth and $H_3(X') = H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. Indeed, in this case $\tau : E \to C_{m-1}$ generates a morphism $\tau_\nu : E \to C^\nu_{m-1}$ where C^ν_{m-1} is a normalization of C_{m-1}. Each fiber $\tau_\nu^{-1}(z)$ consists of one connected component (since otherwise one can easily check that the defining ideal of any of these connected components in $\tau_\nu^{-1}(z)$ is not principal, i.e. E is not a UFD). By Lemma 3.2 (3) E is naturally homeomorphic to $\mathbf{C} \times C^\nu_{m-1}$ and the first claim of Lemma 3.3 holds automatically. In order to have the second claim it is enough to require that the Euler characteristics of E is 1. Then C^ν_{m-1} has Euler characteristics 1 whence it is isomorphic to $\mathbf{C}$.

(2) Furthermore, one may require instead of that condition that X' is smooth, $H_3(X' \setminus E) = 0$, and E has at most isolated singularities (and Euler characteristics 1). Then that E is a topological manifold. The second homology group of this manifold is nontrivial in the case when $\tau^{-1}(z)$ has more then one connected component for some $z \in C_{m-1}$. On the other hand the Thom isomorphism and the exact homology sequence of the pair $(X', X' \setminus E)$ imply that this second homology groups is isomorphic

to $H_4(X')$ which is trivial. Hence $\tau^{-1}(z)$ consists one connected component for each $z \in C_{m-1}$ which yields again Lemma 3.3.

3.2. The Reduced Center.

We shall describe some condition under which the reduced center of an affine modification is a complete intersection.

Lemma 3.5. *Let C be a closed reduced irreducible curve in $\mathbf{C}^n$ and let $\bar{x} = (x_1, \ldots, x_n)$ be a coordinate system on $\mathbf{C}^n$. Suppose that $D_1 = \mathbf{C} \times C$, that v is a coordinate on the first factor of D_1, and $\theta : D_1 \to C$ is the natural projection.*

(1) Let o be a singular point of C, $\mathcal{V}$ be the germ of C at o, and $\mathcal{H} = \theta^{-1}(\mathcal{V})$. Suppose that a function h is holomorphic everywhere in $\mathcal{H}$ except for a finite number of points, and that h is a polynomial in v over the ring of functions on $\mathcal{V}$. Then h is holomorphic in $\mathcal{H}$.

(2) Let h be a holomorphic function on D_1 whose zero set does not contain fibers of θ and let the zero multiplicity of h at generic points of this zero set be n. Suppose that h is a polynomial in v over the ring of functions on $\mathcal{V}$. Then $h^{1/n}$ is a holomorphic function in D_1.

(3) Let C_1 be a closed reduced irreducible algebraic curve in D_1 such that

(i) the projection $\theta|_{C_1} : C_1 \to C$ is finite;

(ii) for every singular point $o \in C$ there exists $\mathcal{H}$ as in (1) such that in the ring of polynomials in v, whose coefficients are holomorphic functions on $\mathcal{V}$, the defining ideal of $C_1 \cap \mathcal{H}$ is principal.

Then the defining ideal of C_1 in $\mathbf{C}[D_1]$ is the principal ideal generated by an irreducible regular function b on D_1 and, if we treat b as the restriction of a polynomial from $\mathbf{C}^{n+1}$ then

$$b(\bar{x}, v) = v^m + p_{m-1}(\bar{x})v^{m-1} + \cdots + p_0(\bar{x})$$

where each $p_i \in \mathbf{C}[\bar{x}]$.

Proof. The argument is of local analytic nature and, therefore, it is enough to consider the case when the normalization of C is $\mathbf{C}$ (actually, we are only interested in the case when C is contractible). Let $\nu_0 : \mathbf{C} \simeq C^{\nu} \to C$ be a normalization and let t be a coordinate on C^{ν}. Then $\nu = (\nu_0, \mathrm{id}) : \mathbf{C}^2 \simeq \mathbf{C} \times C^{\nu} \to D_1$ is a normalization of D_1 and we can suppose that (v, t) is a coordinate system on this sample of $\mathbf{C}^2$. For (1) put $\gamma = h \circ \nu$. This function is holomorphic in a neighborhood of the v-axis in $\mathbf{C}^2$ by the Riemann theorem about deleting singularities, and it is of form

$$\gamma(v, t) = r_k(t)v^k + r_{k-1}(t)v^{k-1} + \cdots + r_0(t).$$

The fact that h is holomorphic everywhere on $\mathcal{H}$ except for a finite number of points implies that for every fixed $v = v_0$ except for a finite number of values the function $\gamma(v_0, t)$ is contained in the ring generated by the coordinate functions $x_1(t), \ldots, x_n(t)$ of ν_0. This implies that each $r_i(t)$ belongs to this ring whence h is holomorphic in $\mathcal{H}$ which is (1).

For (2) note that the function $h^{1/n}$ is holomorphic everywhere in $\mathcal{H}$ except for possibly points from the finite set $h^{-1}(0) \cap \theta^{-1}(o)$. Hence (1) implies (2).

Put $C_1^\nu = \nu^{-1}(C_1)$. It is the zero fiber of an irreducible polynomial $\beta(v, t)$ on $\mathbf{C}^2$. Note that the projection of C_1^ν to the t-axis is finite since $\theta|_{C_1}$ is finite. Hence we can suppose that $\beta(v, t) = v^m + q_{m-1}(t)v^{m-1} + \cdots + q_0(t)$, i.e. β is monic in v. The function $b = \beta \circ \nu^{-1}$ is rational on D_1 and we are going to show that it is, in fact, regular. It suffices to show that b is holomorphic at each point of D_1 (e.g., see [Ka2]). Let o be a singular point of C. Since ν^{-1} is regular outside lines of form $\theta^{-1}(o) \subset D_1$ it is enough to check that b is holomorphic at the points of $\theta^{-1}(o)$.

Let O be the ring of germs of analytic functions at the origin of $\mathbf{C}^n$ (whose coordinate system is $\bar{x}$). Suppose that h be the generator of the defining ideal of $C_1 \cap \mathcal{H}$ in the ring of holomorphic functions on $\mathcal{H}$ that are polynomials in v. By Cartan's theorems (e.g., see [GuRo, Ch. 8A, th. 18]) we can extend each coefficient of h (as a polynomial in v) to a holomorphic function in a Stein neighborhood of the origin in $\mathbf{C}^n$ whence we can treat h as an element of $O[v]$. Suppose that o is the origin of $\mathbf{C}^n$. Let $o_1, \ldots, o_k$ be the set of all points from C_1 such that $\theta(o_i) = o$ for every i. Let c_i be the v-coordinate of o_i. By the Weierstrass Preparation Theorem [Rem, Ch. 1, Th. 1.4] there exists a unique Weierstrass polynomial $\omega_1 \in O[v]$ such that $h = \omega_1(\bar{x}, v - c_1)e_1$ where $e_1 \in O[v]$ does not vanish at o_1. Applying this theorem again we see that there exists a unique Weierstrass polynomial $\omega_2 \in O[v]$ such that $e_1 = \omega_2(\bar{x}, v - c_2)e_2$ where $e_2 \in O[v]$ does not vanish at o_1 and o_2. Hence $h = \omega_1(\bar{x}, v - c_1)\omega_2(\bar{x}, v - c_2)e_2$. Repeating this process we get by induction that $h = \omega e$ where $\omega \in O[v]$ is a monic polynomial, whose zeros on the v-axis are $o_1, \ldots, o_k$, and $e \in O[v]$ does not vanish at o_i for each i (which implies that $e|_\mathcal{H}$ is invertible). Thus $\omega|_\mathcal{H}$ generates the same principal ideal as h. Therefore, we can suppose from the beginning that h is a monic polynomial in v.

Hence $\gamma = h \circ \nu$ is monic as a polynomial in v over the ring of germs of analytic functions at the finite set $\nu_0^{-1}(o) \in \mathbf{C}$. Note that $\gamma = \beta\alpha$ where α does not vanish since the zero multiplicity of γ and β at generic points C_1^ν is 1. Hence since α is a rational function in v it is constant on each line parallel to the v-axis. Furthermore, this constant is 1 since both γ and β are monic (look at the quotient γ/β as v

approaches ∞ along any of these lines). Thus $\beta = \gamma$ whence b coincides with h in $\mathcal{H}$ and, therefore, b is holomorphic. □

Proposition 3.1. *Let the assumptions of Convention 2.2 and Proposition 2.15 hold. Suppose that* $\dim X = 3$, $m \geq 2$ *where* m *is from Proposition 2.15, and either*

(α) X' *is smooth, and* $H_3(X') = H_2(X' \setminus E) = H_3(X' \setminus E) = 0$ *, or*

(β) E *is a UFD and its Euler characteristics is 1.*

Suppose also that $D_1 = E_1$ *is isomorphic to* $\mathbf{C} \times C$ *(i.e.* C *is a curve and it is the reduced center not only of* σ *but of* $\sigma_1 = \delta_1$ *as well) and* $\theta : C_1 \to C$ *is finite where* $\theta = \sigma_1|_{C_1}$. *Let* X *be a local holomorphic UFD with respect to* $A \hookrightarrow A'$ *(see Definition 2.10).*

Then C_1 *is closed in* D_1 *(i.e.* C_1 *is also the reduced center of* ρ_1 *), its defining ideal in* $\mathbf{C}[D_1]$ *is principal, and* C *is contractible.*

Proof. Put $\tau = \rho_{m-1}|_E$ and suppose that L^z be as in Lemma 3.3 and Remark 3.1 where $z \in C_{m-1}$. Let L_i be one the components of L^z. Consider a germ $\mathcal{S}'_i$ of a smooth analytic surface transversal to both E and L_i at a generic point z' of L_i, and consider $\mathcal{C}_j, \mathcal{E}_j, \Gamma_i^j$ as in Lemma 3.3. That is, the germ $\mathcal{S}'_i \cap E$ coincides with $\bigcup_j \Gamma_i^j$. Let $\mathcal{S}_i$ be the germ $\sigma(\mathcal{S}'_i)$ of a surface in X_{m-1} at $z = \sigma(z')$ (indeed, this is the germ of a surface by Lemma 2.2). Then $\mathcal{S}_i \cap D_{m-1}$ consists of components $\mathcal{C}_j$ as above.

Let m_i^j be as before Lemma 3.4. Then the preimage of a generic point $x \in \mathcal{C}_j$ under $\sigma|_{\mathcal{S}'_i}$ consists of m_i^j points. Let x' be one of these points. Recall that $\mathcal{S}'_i$ is smooth at x' and transversal at x' to the exceptional divisor E. Furthermore, we can suppose that $\mathcal{S}'_i$ is transversal to the fiber of σ through x'. Since σ can be viewed as a usual monoidal transformation in a neighborhood of a generic point $x \in C_{m-1}$ (by Remark 2.8) the last fact implies that the germ of $\mathcal{S}_i$ at x consists of m_i^j irreducible smooth components each of which meets D_{m-1} transversally.

Let $\gamma = \sigma_{m-1} \circ \cdots \circ \sigma_2$. Put $\mathcal{S}_i^1 = \gamma(\mathcal{S}_i)$ and let $\mathcal{G}_l$ be the image of $\mathcal{C}_j$ under γ (that is, $\mathcal{G}_l$ is an irreducible component of the germ of C_1 at $z_1 = \gamma(z) = \rho_1(z')$). Suppose that the mapping $\gamma|_{\mathcal{C}_j} : \mathcal{C}_j \to \mathcal{G}_l$ is n_j-sheeted. Let R_l be the set of natural j's such that $\gamma(\mathcal{C}_j) = \mathcal{G}_l$. Since γ is a composition a basic modifications which are nothing but usual monoidal transformations over generic point $x_1 = \gamma(x)$ of $\mathcal{G}_l$ one can see that the germ of $\mathcal{S}_i^1$ at x_1 consists of $\sum_{j \in R_l} n_j m_i^j$ smooth irreducible components which meet D_1 transversally at x_1. By the assumption of this Proposition X is a local holomorphic UFD with respect to $A \hookrightarrow A'$ whence Proposition 2.14 implies the defining ideal of $\mathcal{S}_i^1$ is generated by the germ h_i of a holomorphic function. Since

the zero multiplicity of h_i at smooth points of $\mathcal{S}_i^1$ is 1 we see that the zero multiplicity of $h_i|_{D_1}$ at x_1 is $\sum_{j \in R_l} n_j m_i^j$.

Let L^z consists of k components $L_1, \ldots, L_k$. Since we have an isomorphism $H_0(L^z) \simeq H_1(\mathcal{C}^z)$ by Lemma 3.3, j changes from 1 to k and the matrix (m_i^j) is invertible. Hence there exist integers $s_1, \ldots, s_k$ such that the germ of the function $h = (h_1|_{D^0})^{s_1} \cdots (h_k|_{D^0})^{s_k}$ has the zero multiplicity n_1 at generic points of $\mathcal{G}_1$ and the zero multiplicity 0 at generic points of the other irreducible component of the germ of C_1 at z_1 (i.e. h is different from zero or ∞ at generic points of these other components). Let $\mathcal{D}_1$ be the germ of D_1 at z_1 and let $\mathcal{D}^1, \mathcal{D}^2, \ldots$ be its irreducible components. Since the restriction of h to $\mathcal{D}^i$ is well-defined everywhere on $\mathcal{D}^i$ except may be for z_1 we see that $h|_{\mathcal{D}^i}$ becomes homomorphic function after normalization in virtue of the Riemann theorem about deleting singularities. In particular, it is continuous at z_1 . Furthermore, if $\mathcal{G}_1 \subset \mathcal{D}^1$ then $h|_{\mathcal{D}^1}$ vanishes at z_1 . Since $\mathcal{D}^1 \cap \mathcal{D}^2$ is a curve which contains z_1 we see that $h|_{\mathcal{D}^2}$ vanishes also at z_1 whence the set of zeros of $h|_{\mathcal{D}^2}$ is the germ of a curve, and this germ can be only $\mathcal{G}_1$. But it cannot be true since then $\mathcal{G}_1$ is not contained in the set of double points of D_1 (recall that the morphism $\theta : C_1 \to C$ is not constant). Thus $\mathcal{D}_1$ is irreducible whence the geometrical center C_0 of σ has no double points. Since C_0 admits a non-constant morphism from C_{m-1} and by Lemma 3.3 the normalization of C_{m-1} is $\mathbf{C}$ this implies that C_0 is contractible and coincides with the reduced center C of σ .

Since C_1 admits a non-constant morphism from C_{m-1} it is a once punctured curve and, therefore, it is closed in D_1 . Note that h is holomorphic everywhere on $\mathcal{D}_1$ except, may be, z_1 . By Lemma 3.5 h is holomorphic on $\mathcal{D}_1$. Consider the function e on $\mathcal{D}_1$ such that $e^{n_1} = h$. By Lemma 3.5 it is holomorphic. Recall that the domain of h_1 includes the strip $\sigma_1^{-1}(\mathcal{V}) \simeq \mathbf{C} \times \mathcal{V}$ where $\mathcal{V}$ is the germ of C at $\sigma(z')$ (see Remark 2.9). Hence e is defined in this strip. Let v be a coordinate on the first factor of the strip. For every holomorphic function e_1 on this strip, which is polynomial in v and which vanishes on $\mathcal{G}_1$, the quotient e_1/e is again holomorphic by Lemma 3.5. Therefore, e is the generator of the defining ideal of $\mathcal{G}_1$. Hence the defining ideal of the germ of C_1 at z_1 is principal. Note that $\theta : C_1 \to C$ is a finite morphism since C_1 is a once puncture curve. By Lemma 3.5 the defining ideal of C_1 in the ring of regular functions on D_1 is principal. $\qquad \square$

This concludes the proof of Theorem 2.3.

4. Applications of the Decomposition

4.1. **The proof of Miyanishi's theorem.** We shall reduce first the Miyanishi theorem to a problem about affine modifications.

Lemma 4.1. *Let X' be an affine algebraic variety of dimension 3 such that X' is a UFD and there exists a Zariski open subset Z of X' which is a cylinder over a smooth curve U.*

(i) Then U is a Zariski open subset of $\mathbf{C}$ and the natural projection $p_0 : Z \to U$ can be extended to a regular function $p : X' \to \mathbf{C}$ whose general fibers are still isomorphic to $\mathbf{C}^2$.

(ii) Furthermore, let x, y, z be coordinates on $X = \mathbf{C}^3$. Then there exists an affine modification $\sigma : X' \to X$ such that its coordinate form is $\sigma = (p, p_1, p_2)$ and the divisor of this modification coincides with the zeros of some polynomial $f(x)$ on $\mathbf{C}^3$.

Proof. Let $\bar{F}_c$ be the closure of the fiber $F_c = \{p_0 = c\} \subset Z$ in X' (where $c \in U$). Assume that $\bar{F}_c \cap \bar{F}_{c'} \neq \emptyset$ for some $c \neq c' \in U$. Since X' is a UFD there exists a regular function g on X' whose zero set coincides with the divisor $\bar{F}_{c'}$. Thus the zero locus of $g|_{\bar{F}_c}$ is $\bar{F}_c \cap \bar{F}_{c'}$. On the other hand $g|_{\bar{F}_c}$ is nowhere zero on $\bar{F}_c \setminus (\bar{F}_c \cap \bar{F}_{c'}) \supset F_c \simeq \mathbf{C}^2$ whence this function must be a nonzero constant on F_c and, therefore, $\bar{F}_c$. Contradiction. Thus $\bar{F}_c \cap \bar{F}_{c'} = \emptyset$ for every $c' \neq c \in U$.

Assume $F_c \simeq \mathbf{C}^2$ if different from $\bar{F}_c$. Assume that one of the irreducible components of $\bar{F}_c \setminus F_c$ is a point. Then a normalization G of $\bar{F}_c$ contains a sample of $\mathbf{C}^2$ and one of the irreducible components of $G \setminus \mathbf{C}^2$ is also a point o. By the theorem about deleting singularities of holomorphic functions in codimension 2 for normal complex spaces [Rem, Ch. 13] every holomorphic function on $\mathbf{C}^2$ can be extended to this point o whence $\mathbf{C}^2$ is not Stein. Contradiction. Thus $\bar{F}_c \setminus F_c$ is a curve. Since $\bar{F}_c \cap \bar{F}_{c'} = \emptyset$ we see that the closure of $\bigcup_{c \in U} (\bar{F}_c \setminus F_c)$ is a divisor in X'. Since X' is a UFD there exists a regular function h on X' whose zero set coincides with this divisor. Thus the zero locus of $h|_{\bar{F}_c}$ is $\bar{F}_c \setminus F_c$ and we get a contradiction in the same way we did for function g. Hence $F_c = \bar{F}_c$.

This implies that $p_0 : Z \to U$ can be extended to continuous map p from X' to the completion $\bar{U}$ of U, and $p^{-1}(U) = Z$. In particular, general fibers of p are isomorphic to $\mathbf{C}^2$. Since X' is a UFD p must be holomorphic [Rem, Ch. 13] and, therefore, regular (e.g., see [Ka2]).

Since X' is a UFD (i.e. every effective divisor is the zero divisor of some regular function on X') we see that the Zariski open subset Z of X' is also a UFD whence U is a UFD. This implies that U is rational, i.e. its completion is $\bar{U} = \mathbf{P}^1$.

Show that $p : X' \to \mathbf{P}^1$ is not surjective. Assume the contrary. Let $X_0 = p^{-1}(\mathbf{C})$ and $q = p|_{X_0}$. We can suppose that $Z \subset X_0$, i.e. $U \subset \mathbf{C}$. Extend the isomorphism

$Z \simeq U \times \mathbf{C}^2 \subset \mathbf{C} \times \mathbf{C}^2$ to a rational map from X_0 to $\mathbf{C}^3$ (with coordinate x, y, z) and then multiply the two last coordinates by polynomials in q to make this mapping regular. We obtain a birational morphism $\sigma : X_0 \to \mathbf{C}^3$. This is an affine modification by Theorem 2.1. It is clear that $q = x \circ \sigma$ and the divisor of this modification is given by the zeros of some polynomial $f(x)$ in x . Since $q : X_0 \to \mathbf{C}$ is surjective Proposition 2.5 implies that every invertible function on X_0 is of form $h \circ \sigma$ where h is invertible on $\mathbf{C}^3$. Therefore, each invertible function on X_0 is constant. On the other hand the divisor $p^{-1}(\infty)$ is the zero divisor of some function g on X' since X' is a UFD. Hence $g|_{X_0}$ is invertible and non-constant. Contradiction. Thus one can suppose that $p = q$ and $X' = X_0$ which concludes the proof. □

Now we shall consider the case when the polynomial $f(x)$ constructed in the previous Lemma coincides with x^n .

Lemma 4.2. *Let A be the polynomial ring $\mathbf{C}[x, y, z]$ in three variables x, y , and z , let $f = x^n$, and let $A \hookrightarrow A'$ be an affine modification. Suppose that $q(x)$ is a polynomial in x such that $q(0) \neq 0$. Let $B = A[1/q], J = I[1/q]$, and let $B \hookrightarrow B'$ be an affine modification with locus (J, f) . Suppose that B' is a UFD and one of the following conditions hold*

(α) E is non-empty, X' is smooth, and $H_3(X') = 0$;

(β) E is a UFD and and its Euler characteristics is $e(E) = 1$.

Let E have at most isolated singularities. Then A' is also a polynomial ring $\mathbf{C}[x, u, v]$ in variables x, u , and v .

Proof. Let $\delta : Y' \to Y = \mathbf{C}^3 \setminus \{q^{-1}(0)\}$ be the affine modification of the varieties which corresponds to the modification $B \hookrightarrow B'$ with locus (J, f) . Since affine modifications commute with localizations in the sense of Proposition 2.1 we see that E is also the exceptional divisor of this modification δ . If $B' \neq B$ we can suppose that the reduced center of this modification is at least of codimension 2 in Y . (Indeed, otherwise each element $h \in J$ must vanish on the plane $\{x = 0\} \subset \mathbf{C}^3$ whence $h = xh_1$. We can replace the locus (J, x^n) by the locus $(J/x, x^{n-1})$. After several replacement like this one we shall obtain an element of J which does not vanish on the plane.)

Show that Theorem 2.3 is applicable which is clear if condition (β) holds. Let $T = \mathbf{C}^3 \setminus \{xq(x) = 0\}$ and $Z = \mathbf{C}^3 \setminus \{x = 0\}$. Note that $H_i(Z) = H_i(T) = 0$ for $i \geq 2$ and T is isomorphic to $Y' \setminus E$. Hence $H_2(Y' \setminus E) = H_3(Y' \setminus E) = 0$. We can glue Y' and Z along $T \simeq Y' \setminus E$. Then we have $X' = Y' \bigcup_T Z$. The

Mayer-Viertoris theorem implies that $H_3(X') = H_3(Y') = 0$. Hence condition (α) in this Lemma implies condition (α) in Theorem 2.3.

By Theorem 2.3 we have now a sequence of basic modifications

$$B = B_0 \hookrightarrow B_1 \hookrightarrow \ldots \hookrightarrow B_m = B'$$

which corresponds to the sequence of morphisms

$$Y' = Y_m \xrightarrow{\delta_m} Y_{m-1} \to \ldots \to Y_1 \xrightarrow{\delta_1} Y = \mathbf{C}^3 \setminus \{q^{-1}(0)\}.$$

The natural embeddings $B_i \hookrightarrow B'$ are affine modifications which generate morphisms $\theta_i : Y' \to Y_i$. Let $C_i = \theta_i(E)$. By Theorem 2.3 each C_i is either a point or a smooth contractible irreducible curve which is a connected component of the geometrical center of δ_i.

Our first aim is to show that B' is a localization of the polynomial ring $\mathbf{C}[x, u, v]$ with respect to to the multiplicative system $\{q^n(x) \,|\, n \in \mathbf{N}\}$ where $u, v \in A'$. Suppose that $C_0 = C$ is a point (say, the origin $o = \{x = y = z = 0\}$). Let M be the maximal ideal in B that vanishes at o. By Theorem 2.3 $B_1 = B[M/x]$. Hence $B_1 = A_1[1/q]$ where A_1 is the polynomial ring $\mathbf{C}[x, y/x, z/x]$ in three variables. Suppose that j is the first number for which C_j and, therefore, every C_k with $k > j$ are curves (recall that $\delta_i|_{C_i} : C_i \to C_{i-1}$ must be surjective by Proposition 2.15). By induction we can suppose that $B_j = A_j[1/q]$ where A_j is a polynomial ring $\mathbf{C}[x, \xi, \zeta]$ (in particular, the divisor D_j of θ_j and δ_{j+1} is the $\xi\zeta$ -plane).

By Theorem 2.3 C_j is isomorphic to $\mathbf{C}$ and by the Abhyankar-Moh-Suzuki theorem one can assume that it is given by $x = \xi = 0$. Let I_j be the ideal generated by x and ξ. By Theorem 2.3 $B_{j+1} = B_j[I_j/x]$. Then by Proposition 2.1 $B_{j+1} = A_{j+1}[1/q]$ where A_{j+1} is the polynomial ring $\mathbf{C}[x, \xi/x, \zeta]$ in three variables. Hence our first aim can be achieved by induction.

By Proposition 2.11 the projection $Y' \to Y$ generates isomorphisms of the homology groups and the fundamental groups. As we mentioned in the beginning of the proof $X' = Y' \bigcup_T Z$ and, similarly, $\mathbf{C}^3 = Y \bigcup_T Z$. The Mayer-Viertoris and van Kampen-Seifert theorems imply that the homology groups and the fundamental group of X' are trivial since this is true for $\mathbf{C}^3$. Hence X' is contractible. Note also that X' is smooth since Y' is smooth. This implies that X' is a UFD (e.g., see [Ka1]). Thus if we put $q(x)$ equal to 1 then the assumptions of this Lemma are true (since we know now that A' is a UFD). Therefore, $B' = A'[1/q] = A' = \mathbf{C}[x, u, v]$. $\qquad\square$

Lemma 4.3. *Let* $A = \mathbf{C}[x, y, z]$, *let* f *is a polynomial in one variable* x, *and let* $A \hookrightarrow A'$ *be an affine modification. Suppose that* A' *is a UFD and that for every*

root c of f the polynomial f − c is not a unit in A′ (or, equivalently the mapping f ∘ σ : X′ → C is surjective). Let one of the following conditions hold

(α) X′ is smooth and $H_3(X') = 0$;

(β) $e(X') = 1$ and every irreducible component of E is a UFD.

If every irreducible component of E has at most isolated singularities (which is automatically true under condition (β)) then X′ is isomorphic to $\mathbf{C}^3$ and $x \circ σ$ is a variable on this sample of $\mathbf{C}^3$.

Proof. Let $f(x) = x^n q(x)$ where $q(0) \neq 0$, $J = I[1/q]$ and $B = A[1/q]$. By Proposition 2.1 $B' = B[J/x^n]$ coincides with $A'[1/q]$. Hence B and B' are UFDs and the exceptional divisor E^0 of the modification $B \hookrightarrow B'$ is not empty by the assumption on $f \circ σ$. It is irreducible by Proposition 2.4. This makes Lemma 4.2 applicable to this modification under condition (α). Show that the same is true under (β). By Proposition 2.15 we can present $B \hookrightarrow B'$ as a composition of basic modifications. Hence E^0 is homeomorphic to $C^0 \times \mathbf{C}^s$ where C^0 is a point (see Lemma 2.4 and Remark 2.10) or an irreducible curve (see Lemma 3.2 and Remark 3.1) and $s = 2$ or 1 respectively. That is, $e(E_0) \leq 1$ in any case. Let D_0 be the coordinate plane $x = 0$. Then by the additivity of Euler characteristics [Du] $e(X')$ differs from $e(X) = e(\mathbf{C}^3) = 1$ by the sum of terms of form $e(E_0) - e(D_0)$ (these terms should be considered for each root of f). Since $e(X') = 1$ we see that $e(E_0) = e(D_0) = 1$ which makes Lemma 4.2 applicable.

Suppose that L is the ideal in A generated by I and x^n , i.e. $I[1/q] = L[1/q] = J$. By Lemma 4.2 $B' = A^1[1/q]$ where $A^1 = A[L/x^n] = \mathbf{C}[x, u, v]$ is a polynomial ring in three variables. Let K be the ideal in A^1 generated by I/x^n . By Theorem 2.1 (3) $A' = A^1[K/q]$. Now the induction by the degree of f implies implies the desired conclusion. □

Remark 4.1. In fact the assumption that A' is a UFD can be replaced by a weaker one. Namely, one can assume only that for every root c of f there exists a polynomial $r(x)$ with $r(c) \neq 0$ such that the localization $A[1/r]$ is a UFD.

Lemmas 4.1 and 4.3 imply

Miyanishi's Theorem (Lemma I). *Let X′ be an affine algebraic variety of dimension 3 such that X′ is a UFD, all invertible functions on X′ are constants, and*

(1) the Euler characteristics of X′ is $e(X') = 1$;

(2′) there exists a Zariski open subset Z of X′ which is a $\mathbf{C}^2$ -cylinder over a curve U (i.e. Z is isomorphic to the $\mathbf{C}^2 \times U$);

(3) each irreducible component of $X' \setminus Z$ is a UFD.

Then X' is isomorphic to $\mathbf{C}^3$.

(4) Furthermore, the curve U is a Zariski open subset of $\mathbf{C}$, the natural projection from Z to U can be extended to a regular function on X' , and this function is a variable.

The statement of this Lemma remains true if conditions (1) and (3) are replaced by

(1') X' is smooth and $H_3(X') = 0$;

(3') each irreducible component of $X' \setminus Z$ has at most isolated singularities.

4.2. **How to present X' as a closed algebraic subvariety of $\mathbf{C}^N$.** In this section we shall study X' which satisfies assumptions ($1'$) and ($2'$) of Lemma I, but we do not require ($3'$). We want to present X' explicitly as a closed affine algebraic subvariety of some Euclidean space $\mathbf{C}^N$.

Lemma 4.4. Let $A = \mathbf{C}[x, y, z]$ and $f(x) = x^n$. Let $A \hookrightarrow A'$ be an affine modification. Suppose that X' is a smooth UFD, the only invertible functions on X' are constants, and $H_3(X') = 0$. Then

(1) either X' is isomorphic to $\mathbf{C}^3$ or at least

(2) X' can be viewed as the subvariety of $\mathbf{C}^{3+m}$ given by polynomial equations

$$xv_1 - q_0(y, z) = 0$$
$$xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0$$
$$\dots$$
$$xv_m - v_{m-1}^{n_{m-1}} + q_{m-1}(y, z, v_1, \dots, v_{m-1}) = 0$$

where the usual degree of q_j with respect to v_i is less than n_i for every $i = 1, \dots, j$. Furthermore, one can suppose that $q_0(y, z) = y^k - z^l$ where $(k, l) = 1, k > l \geq 2$, and $m \geq 2$.

Proof. Since $X' \setminus E \simeq \mathbf{C}^3 \setminus \{x = 0\}$ we have $H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. By Theorem 2.3 the modification $\sigma : X' \to X$ is a composition of basic modifications

$$X' = X_m \overset{\sigma_m}{\to} X_{m-1} \to \dots \to X_1 \overset{\sigma_1}{\to} X.$$

Let $A_j = \mathbf{C}[X_j]$ and C_j be as in Convention 2.2 (3). By Theorem 2.3 each C_j is either a point or an irreducible contractible curve. If C_0 is a point then, as it was shown in the proof of Lemma 4.2, $A_1 \simeq \mathbf{C}[x, y/x, z/x]$ whence $X_1 \simeq \mathbf{C}^3$. Therefore, we can suppose that C_0 is a curve whence each C_i is a curve since $\sigma|_{C_i} : C_i \to C_{i-1}$ is surjective by Proposition 2.15. If C_0 is a smooth curve, i.e. $C_0 \simeq \mathbf{C}$ then it was shown in the proof of Lemma 4.2 that we can suppose that $A_1 \simeq \mathbf{C}[x, y/x, z]$, i.e.

$X_1 \simeq \mathbf{C}^3$. Therefore, we consider the case when C_0 is not smooth. By the Lin-Zaidenberg theorem [LiZa] one can assume that the equations of this curve in $\mathbf{C}^3$ are $x = y^k - z^l = 0$ where $(k, l) = 1$ and $k > l \geq 2$. Let I_1 be the ideal in A generated by x and $y^k - z^l$. By Theorem 2.3 $A_1 = A[I_1/x]$. By Theorem 2.2 this implies that $A_1 = \mathbf{C}[x, y, z, (y^k - z^l)/x]$ and X_1 is the hypersurface in $\mathbf{C}^4$ with coordinates (x, y, z, v_1) given by

$$xv_1 = q_0(y, z) := y^k - z^l.$$

The exceptional divisor E_1 is σ_1 is the intersection of this hypersurface with the hyperplane $x = 0$. By Theorem 2.3 and by Lemma 3.5 C_1 is the zero fiber of a regular function on E_1 which is of form $v_1^{n_1} + q_1(y, z, v_1)$ where the usual degree of q_1 with respect to v_1 is at most $n_1 - 1$. Let I_2 be the ideal in A_1 generated by x and $v_1^{n_1} + q_1(y, z, v_1)$. By Theorem 2.3 $A_2 = A_1[I_2/x]$. Therefore, by Theorem 2.2 X_2 may be viewed as the subvariety of $\mathbf{C}^5$ (with coordinates (x, y, z, v_1, v_2)) given by the equations

$$xv_1 - q_0(y, z) = 0$$
$$xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0.$$

Repeating the above argument we see that X' can be viewed as the subvariety of $\mathbf{C}^{3+m}$ given by the equations

$$xv_1 - q_0(y, z) = 0$$
$$xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0$$
$$\cdots$$
$$xv_m - v_{m-1}^{n_{m-1}} + q_{m-1}(y, z, v_1, \ldots, v_{m-1}) = 0$$

where the usual degree of q_j with respect to v_i is less than n_i for every $i = 1, \ldots, j$.

In order to check that $m > 1$ when X' is smooth it is enough to note that X_1 is singular at the origin. $\qquad\square$

Proposition 4.1. *Let $A = \mathbf{C}[x, y, z]$ and f be a polynomial in x. Suppose that $A \hookrightarrow A'$ is an affine modification. Let $c_0, c_1, \ldots$ be the roots of f. Suppose that X' is a smooth UFD, the only invertible functions on X' are constants, and $H_3(X') = 0$. Then*

(1) either X' is isomorphic to $\mathbf{C}^3$ or at least there exists a root of f (say c_0 and assume that $c_0 = 0$) such that

(2) X' can be viewed as the subvariety of $\mathbf{C}^N$ given by a system of polynomial equations

$$xv_1 - q_0(y,z) = 0$$
$$xv_2 - v_1^{n_1} + q_1(y,z,v_1) = 0$$
$$\ldots$$
$$xv_m - v_{m-1}^{n_{m-1}} + q_{m-1}(y,z,v_1,\ldots,v_{m-1}) = 0$$
$$(x-c_1)u_{1,1} - r_{1,0}(y,z) = 0$$
$$(x-c_1)u_{1,2} - u_{1,1}^{n_{1,1}} + r_{1,1}(y,z,u_{1,1}) = 0$$
$$\ldots$$
$$(x-c_1)u_{1,m_1} - u_{1,m_1-1}^{n_{1,m_1-1}} + r_{1,m_1-1}(y,z,u_{1,1},\ldots,u_{1,m_1-1}) = 0$$
$$(x-c_2)u_{2,1} - r_{2,0}(y,z) = 0$$
$$\ldots$$

where $q_0(y,z) = y^k - z^l$, $(k,l) = 1, k > l \geq 2$ and $m > 1$. Furthermore,

-the usual degree of q_j with respect to v_i is less than n_i for every $i = 1,\ldots,j$ and;

- $r_{s,j}$ are polynomials such that the usual degree of $r_{s,j}$ with respect to $u_{s,i}$ is less than $n_{s,i}$ for every $i = 1,\ldots,j$.

The variety X' is irreducible, it is a complete intersection in $\mathbf{C}^N$, i.e. the ideal I' of polynomials that vanish on X' is generated by the left-hand sides of this system.

Proof. We consider the case when f has two roots 0 and 1 (i.e. $f(x) = x^n(x-1)^s$) since the general case differs only by more complicated notation. Consider $X^1 = \operatorname{spec} A[I_1/x^n]$ where I_1 is generated by I and x^n . By Lemma 4.4 X^1 coincides with the zero set of the system

$$xv_1 - q_0(y,z) = 0$$
$$xv_2 - v_1^{n_1} + q_1(y,z,v_1) = 0$$
$$\ldots$$
$$xv_m - v_{m-1}^{n_{m-1}} + q_{m-1}(y,z,v_1,\ldots,v_{m-1}) = 0$$

where the usual degree of q_j with respect to v_i is less than n_i for every $i = 1,\ldots,j$ and $m > 1$. Let $X^2 := \operatorname{spec} A[I_2/(x-1)^s]$ where I_2 is generated by I and $(x-1)^s$. By Lemma 4.4 X^2 can be given by the zeros of the system

$$(x-1)u_1 - r_0(y,z) = 0$$
$$(x-1)u_2 - u_1^{1,n_1} + r_1(y,z,u_1) = 0$$
$$\ldots$$
$$(x-1)u_{m_1} - u_{m_1-1}^{1,n_{m_1-1}} + r_{m_1-1}(y,z,u_1,\ldots,u_{m_1-1}) = 0$$

46

where r_j are polynomials such that the usual degree of r_j with respect to u_i is less than n_i for every $i = 1, \ldots, j$. By Remark 2.6 we see that X' is isomorphic to the common zeros of the system

$$xv_1 - q_0(y, z) = 0$$
$$xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0$$
$$\cdots$$
$$xv_m - v_{m-1}^{n_{m-1}} + q_{m-1}(y, z, v_1, \ldots, v_{m-1}) = 0$$
$$(x - 1)u_1 - r_0(y, z) = 0$$
$$(x - 1)u_2 - u_1^{1,n_1} + r_1(y, z, u_1) = 0$$
$$\cdots$$
$$(x - 1)u_{m_1} - u_{m_1-1}^{1,n_{m_1}-1} + r_{m_1-1}(y, z, u_1, \ldots, u_{m_1-1}) = 0.$$

In order to prove the last statement consider X_1 given in $\mathbf{C}^4$ by the equation

$$xv_1 - q_0(y, z) = 0.$$

Clearly, X_1 is a complete intersection and it is irreducible. Consider X_2 given in $\mathbf{C}^5$ by the equations

$$xv_1 - q_0(y, z) = 0$$
$$xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0.$$

Let $A_i = \mathbf{C}[X_i]$. Note that $A_2 = A_1[v_1^{n_1}/x]$, is a Davis modification. In particular, X_2 is irreducible. By Theorem 2.2 the ideal of polynomials that vanishes on X_2 is generated by the left-hand sides of these two equations. Now the induction implies the similar conclusion about X'. $\qquad\square$

Suppose that X' satisfies the assumptions ($1'$) and ($2'$) of Lemma I. By Lemma 4.1 this X' satisfies the assumption of Proposition 4.1 whence we have

Corollary 4.1. *Suppose that X' satisfies the assumptions ($1'$) and ($2'$) of Lemma I. Then X' satisfies also either (1) or (2) from Proposition 4.1.*

5. The Makar-Limanov Invariant

5.1. **General Facts about Locally Nilpotent Derivations.** Recall the following

Definition 5.1. A derivation ∂ on A is a linear endomorphism which satisfies the Leibniz rule, i.e. $\partial(ab) = a\partial(b) + b\partial(a)$. Two derivations are called *equivalent* if they have the same kernel. A derivation ∂ is called locally nilpotent if for each $a \in A$ there exists an $k = k(a)$ such that $\partial^k(a) = 0$.

Every locally nilpotent derivation defines a degree function $\deg_\partial$ on the domain A with natural values (e.g., see [FLN]) given by the formula $\deg_\partial(a) = \max\{k \mid \partial^k(a) \neq 0\}$ for every nonzero $a \in A$. The first five statements of the following theorem can be found in [M-L1], [M-L2], [KaM-L1], and [KaM-L2].

Theorem 5.1. *Let ∂ be a nonzero locally nilpotent derivation of A.*

(1) A has transcendence degree one over $\mathrm{Ker}\,\partial$. The field $\mathrm{Frac}(A)$ of fractions of A is a purely transcendental extension of $\mathrm{Frac}(\mathrm{Ker}\,\partial)$, and $\mathrm{Ker}\,\partial$ is algebraically closed in A.

(2) Let $b \in A$ and $\deg_\partial(b) = 1$. Then for every $a \in A$ such that $\deg_\partial(a) = k$ there exist $a', a_0, a_1, \ldots, a_k \in \mathrm{Ker}\,\partial$ for which $a', a_k \neq 0$ and $a'a = \sum_{j=0}^{k} a_j b^j$.

(3) Every two locally nilpotent derivations ∂ and δ on A are equivalent iff they generate the same degree function. Furthermore, there exist $\alpha, \beta \in A^\partial$ such that $\alpha\partial = \beta\delta$.

(4) Suppose $a_1, a_2 \in A$. Then $a_1 a_2 \in \mathrm{Ker}\,\partial \setminus \{0\}$ implies $a_1, a_2 \in \mathrm{Ker}\,\partial$. In particular, every unit $u \in A$ belongs to $\mathrm{Ker}\,\partial$.

(5) Suppose that $a_1^k + a_2^l \in \mathrm{Ker}\,\partial \setminus \{0\}$ where $k, l \geq 2$ are relatively prime. Then $a_1, a_2 \in \mathrm{Ker}\,\partial$.

(6) (cf. [Za], proof of Lemma 9.3) Let $A = \mathbf{C}[X]$ and let $F = (f_1, \ldots, f_s) : X \to Y \subset \mathbf{C}^s$ and $G : Y \to Z \subset \mathbf{C}^j$ be dominant morphisms of reduced affine algebraic varieties. Put $H = G \circ F = (h_1, \ldots, h_j) : X \to Z$. Suppose that for generic point $\xi \in Z$ there exists a (Zariski) dense subset T_ξ of $G^{-1}(\xi)$ such that the image of any non-constant morphism from $\mathbf{C}$ to $G^{-1}(\xi)$ does not meet T_ξ. Then the fact that $h_1, \ldots, h_j \in A^\partial$ implies that $f_1, \ldots, f_s \in A^\partial$.

Proof of (6). Consider the $\mathbf{C}_+$-action on X generated by ∂. Choose a generic point $\xi \in Z$ and any point $\zeta \in H^{-1}(\xi)$. Consider the orbit O_ζ of this point ζ under the action. Since $h_1, \ldots, h_j \in A^\partial$ the subvariety $H^{-1}(\xi)$ is invariant under the action which means that O_ζ is contained in $H^{-1}(\xi)$. If $F(O_\zeta)$ is not a point it cannot meet T_ξ. But $F^{-1}(T_\xi)$ is (Zariski) dense in $H^{-1}(\xi)$. Thus there is a point θ from $F^{-1}(T_\xi)$ in every (Zariski) neighborhood of ζ. Since O_θ is a point O_ζ is also a point whence each orbit of the action is contained in a fiber of $F = (f_1, \ldots, f_m)$. This implies the desired conclusion. $\square$

Remark 5.1. In fact, (6) implies (4) and (5). Indeed, let $a_1, a_2 \in A$ and either $a_1 a_2 \in \mathrm{Ker}\,\partial \setminus \{0\}$ or $a_1^k + a_2^l \in \mathrm{Ker}\,\partial \setminus \{0\}$ where $k > l \geq 2$ and $(k, l) = 1$. Then $a_1, a_2 \in A^\partial$. In order to see this we apply (6) in the case when $F = (a_1, a_2) : X \to Y$

where Y is the closure of $F(X)$ in $\mathbf{C}^2$, and $G : Y \to Z := G(Y) \subset \mathbf{C}$ is given either by $(x, y) \to xy$ or $(x, y) \to x^k + y^l$.

More generally, we have the following result which will not be used in this paper.

Corollary 5.1. *Let ∂ be a nonzero locally nilpotent derivation of A, and let $a_1, a_2 \in A$. Suppose that $p \in \mathbf{C}[x, y]$ is a non-constant polynomial which is not equivalent to a linear one. Let $p(a, b) \in \operatorname{Ker} \partial$ and let one of the following conditions be true*

(i) $p(a, b) \notin \mathbf{C}$, or

(ii) $p(a, b) = c \in \mathbf{C}$ and the curve $p^{-1}(c) \subset \mathbf{C}^2$ does not contain an irreducible contractible component.

Then a and b belong to $\operatorname{Ker} \partial$.

Consider an ascending filtration $\mathcal{F} = \{F^t A\}$ on A, where $t \in \mathbf{R}$ and $F^t A \subset F^s A$ for $t < s$. Put $F_0^t A = \bigcup_{s < t} F^s A$.

Definition 5.2. Consider the linear space $\operatorname{Gr} A = \oplus_{t \in \mathbf{R}} \operatorname{Gr}{}^t A$ where $\operatorname{Gr}{}^t A = F^t A / F_0^t A$, and introduce the following multiplication on $\operatorname{Gr} A$. Suppose that $f_1 \in F^{t_1} A / F_0^{t_1} A$ and $f_2 \in F^{t_2} A / F_0^{t_2} A$. Put $(f_1 + F_0^{t_1} A)(f_2 + F_0^{t_2} A)$ equal to $f_1 f_2 + F_0^{t_1 + t_2} A$ if $f_1 f_2 \in F^{t_1 + t_2} A \setminus F_0^{t_1 + t_2} A$ and 0 otherwise (of course, the last possibility does not hold in the case when the filtration is generated by a degree function). Extend this multiplication using the distributive law. Then we call $\widehat{A} = \operatorname{Gr} A$ the *associated graded algebra* of the filtered algebra $(A, \mathcal{F})$.

Definition 5.3. Define the mapping $\operatorname{gr} : A \to \operatorname{Gr} A$ by $\operatorname{gr} f = \widehat{f} = f + F_0^t A$ when $f \in F^t A \setminus F_0^t A$. If the filtration is generated by a degree function then this mapping gr is a multiplicative homomorphism.

Lemma 5.1. [KaM-L2] *Suppose that $\mathcal{F}$ is a weight filtration on A (see the definition of a weight filtration on an algebra of regular functions after Lemma 5.2 below). Then for every derivation ∂ on A there exists the smallest $t_0 \in \mathbf{R}$ (which is called the degree of ∂) such that $\partial(F^t A) \subset F^{t+t_0} A$ for every $t \in \mathbf{R}$. Furthermore, there exists $a \in F^t A$ for some t such that $a \in F^{t+t_0} A \setminus F_0^{t+t_0} A$.*

Definition 5.4. Consider the function $\operatorname{def}_\partial : A \setminus 0 \to \mathbf{R} \cup \infty$ given by

$$\operatorname{def}_\partial(a) = d_A(a) - d_A(\partial(a)).$$

Every nonzero $\partial \in \mathrm{LND}(A)$ defines a nonzero $\widehat{\partial} \in \mathrm{LND}(A)$ as follows: $\widehat{\partial}(\widehat{a}) = \widehat{\partial(a)}$ if $\mathrm{def}_\partial(a)$ coincides with the negative degree of ∂, and $\widehat{\partial}(\widehat{a}) = 0$ otherwise. We call $\widehat{\partial}$ the associate locally nilpotent derivation for ∂.

Definition 5.5. The Makar-Limanov invariant of A is

$$\mathrm{ML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Ker}\, \partial$$

where $\mathrm{LND}(A)$ is the set of all locally nilpotent derivations on A. When $\mathrm{ML}(A) = \mathbf{C}$ we call the invariant trivial (this is so when A is the ring of polynomials).

Remark 5.2. For a locally nilpotent derivation ∂ and every $t \in \mathbf{C}$ the mapping $\exp(t\partial) : A \to A$ is an automorphism whence it generates a $\mathbf{C}_+$-action on X [Ren]. When ∂ is nonzero this action is nontrivial. Hence $\mathrm{ML}(A)$ coincides with the subset of A which consists of those regular functions on X that are invariant under any regular $\mathbf{C}_+$-action.

The method of computation of this invariant which we are going to exploit, is based on two ideas.

First, there is no need to consider all the set $\mathrm{LND}(A)$ in the definition of $\mathrm{ML}(A)$. It is enough to consider its subset S such that it contains at least one representative from every equivalence class. Then $\mathrm{ML}(A) = \bigcap_{\partial \in S} \mathrm{Ker}\, \partial$.

Second, one can study $\mathrm{LND}(\widehat{A})$ which may be easier and then one can use the knowledge of $\mathrm{LND}(\widehat{A})$ in order to find all $\partial \in S$. This second step requires a more convenient description of $\widehat{A}$ for some specific filtrations.

5.2. **The associate algebra $\widehat{A}'$.** Let $A' = \mathbf{C}^{[N]}/I'$ where I' is a prime ideal in the ring of polynomials $\mathbf{C}^{[N]}$ in N variables $x_1, \ldots, x_N$. For every $a \in A'$ we denote by $[a]$ the set of polynomials $p \in \mathbf{C}^{[N]}$ such that $p|_{X'} = a$. Each nonzero polynomial p is the sum of nonzero monomials, and the set of this monomials will be denoted by $M(p)$.

Definition 5.6. A *weight degree function* on the polynomial algebra $\mathbf{C}^{[N]}$ is a degree function d such that $d(p) = \max\{d(\mu) \,|\, \mu \in M(p)\}$, where $p \in \mathbf{C}^{[N]}$ is a non-zero polynomial. Clearly, d is uniquely determined by the *weights* $d(x_i) \in \mathbf{R}$, $i = 1, \ldots, N$. A weight degree function d defines a grading $\mathbf{C}^{[N]} = \oplus_{t \in \mathbf{R}} \mathbf{C}_{d,t}^{[N]}$, where $\mathbf{C}_{d,t}^{[N]} \setminus \{0\}$ consists of all the $d-$ homogeneous polynomials of $d-$ degree t. Accordingly, for any $p \in \mathbf{C}^{[N]} \setminus \{0\}$ we have a decomposition $p = h_{t_1} + \ldots + h_{t_k}$ into a sum of

$d-$ homogeneous components h_{t_i} of degree t_i where $t_1 < t_2 < \ldots t_k = d(p)$. We call $\bar{p} := h_{d(p)}$ the *principal component* of p.

Definition 5.7. Let d be a weight degree function on $\mathbf{C}^{[N]}$. For $a \in A' \setminus \{0\}$ set

$$d_{A'}(a) = \inf_{p \in [a]} d(p).$$

Let $\widehat{I}'_d$ be the (graded) ideal of $\mathbf{C}^{[N]}$ generated by the principal components of the elements of I'.

Lemma 5.2. [KaM-L2] *For every nonzero $a \in A'$ we have*

(1) there exists a polynomial $p \in [a]$ such that $\bar{p} \notin \widehat{I}'_d$;

(2) $d_{A'}(a) = d(q)$ for a polynomial $q \in [a]$ iff $\bar{q} \notin \widehat{I}'_d$. In particular, $d_{A'}(a) = \min_{q \in [a]}\{d(q)\}$;

(3) if $\widehat{I}'_d$ is prime then $d_{A'}$ is a degree function on A'.

Note that $F^t A' := \{a \in A' \mid d_{A'}(a) \leq t\}$ where $t \in \mathbf{R}$ gives a filtration on A'. We shall call such a filtration on the algebra A' of regular functions of an affine algebraic variety X' a *weight filtration*. This filtration generates the associate graded algebra $\widehat{A}'_d$ and the mapping $\mathrm{gr}_d : A' \to \widehat{A}'_d$ which is a multiplicative homomorphism in the case when $\widehat{I}'_d$ is prime.

Proposition 5.1. [KaM-L2] *The associated graded algebra is isomorphic to*

$$\widehat{A}'_d \simeq \mathbf{C}^{[N]}/\widehat{I}'_d = \mathbf{C}[\widehat{X}'_d],$$

where $\widehat{X}'_d$ is the affine variety in $\mathbf{C}^N$ defined by the ideal $\widehat{I}'_d$. Furthermore, for every nonzero $a \in A'$ we have $\mathrm{gr}_d(a) = \bar{p}|_{\widehat{X}'_d}$ where $p \in [a]$ and $d(p) = d_{A'}(a)$.

Convention 5.1. Consider the coordinate system $(x, y, z, v_1, \ldots, v_m, u_{1,1}, \ldots, u_{j,i}, \ldots)$ in the space $\mathbf{C}^N$ which appeared in Proposition 4.1. Let $q_0(y, z) = y^k - z^l, m_i, n_{j,i}$ be as in Proposition 4.1. Put $d_x = d(x), d_y = d(y), d_z = d(z), d_i = d(v_i)$ and $d_{j,i} = d(u_{j,i})$ where d is a weight degree function. From now on we are going to study only those weight degree functions on $\mathbf{C}^{[N]}$ that satisfy the following

(1) $k d_y = l d_z$ (in particular, $\bar{q}_0 = q_0 = y^k - z^l$);

(2) $d_1 + d_x = k d_y$, and d_1, d_x are $\mathbf{Q}$-independent (this implies that the elements in the following pairs $(d_x, d_y), (d_x, d_z), (d_1, d_y), (d_1, d_z)$ are $\mathbf{Q}$-independent);

(3) $d_x < 0$ and $d_1 \gg d_y > 0$;

(4) $d_x + d_{i+1} = m_i d_i$ for $i \geq 1$;

51

(5) $d_x + d_{j,i+1} = n_{j,i}d_{j,i}$ for every $j, i \geq 1$.

Proposition 5.2. *Let X' be the zero set of the system of polynomial equations from Proposition 4.1 and $A' = \mathbf{C}[X']$. Then under Convention 5.1 the associate graded algebra $\widehat{A}'_d = \mathbf{C}[\widehat{X}'_d]$ where $\widehat{X}'_d$ is isomorphic to the zero set of the following system*

$$xv_1 - q_0(y, z) = 0$$
$$xv_2 - v_1^{n_1} = 0$$
$$\ldots$$
$$xv_m - v_{m-1}^{n_{m-1}} = 0$$
$$-c_1 u_{1,1} = 0$$
$$-c_1 u_{1,2} - u_{1,1}^{n_{1,1}} = 0$$
$$\ldots$$
$$-c_1 u_{1,m_1} - u_{1,m_1-1}^{n_{1,m_1-1}} = 0$$
$$-c_2 u_{2,1} = 0$$
$$\ldots$$

Furthermore, the defining ideal $\widehat{I}'_d$ of $\widehat{X}'_d$ is prime (i.e., $\widehat{A}'_d$ is a domain) and it is generated by the left-hand sides of the equations above.

Proof. By Proposition 4.1 the ideal I' is generated by the left-hand sides of the equations from that Proposition. The principal components of these left-hand sides coincides with the left-hand side of the system above. For the first equation it follows from Convention 5.1 (1) and (2). For the second equation it follows from Convention 5.1 (3) and the fact that the usual degree of q_1 from Proposition 4.1 with respect to v_1 is at most $n_1 - 1$. Similarly, taking into consideration the assumption on the usual degrees of q_j with respect to v_i and $r_{s,j}$ with respect to $u_{s,i}$ we obtain the claim about principal components.

The polynomial system from this Proposition defines an algebraic variety Y which is irreducible and a complete intersection, i.e. its defining ideal K is generated by the left-hand sides of the equations above. (Indeed, one can apply, for instance, the same argument with the Davis theorem which we used while proving that X' is irreducible and a complete intersection in Proposition 4.1.) Clearly, $K \subset \widehat{I}'_d$. In order to show the reverse inclusion we need to prove that every element of $\widehat{I}'_d$ vanishes on Y.

Consider $\mathbf{C}^{N+1}$ which contains $\mathbf{C}^N$ from Proposition 4.1 as a coordinate N-plane. Suppose that the coordinates are $x, y, z, v_i, u_{i,j}$ (as in Proposition 4.1), and ξ. Consider the subvariety $Z \subset \mathbf{C}^{N+1}$ given by the same system as in Proposition 4.1 and the additional equation $x\xi - 1 = 0$ (one can see that Z is the localization of X'

with respect to the multiplicative system $\{x^n \,|\, n \geq 0\}$). In this system of equations we can replace $xv_1 - q_0(y, z) = 0$ by $v_1 - \xi q_0(y, z) = 0$, $xv_2 - v^{n_1} + q_1(y, z, v_1) = 0$ by $v_2 - \xi(v^{n_1} + q_1(y, z, v_1)) = 0$, etc.. Clearly, the defining ideal J of Z is generated by $x\xi - 1 = 0$ and these replacements. More precisely, every element of J is of form

$$\alpha_0(x\xi - 1) + \alpha_1(v_1 - \xi q_0(y, z)) + \alpha_2(v_2 - \xi(v^{n_1} + q_1(y, z, v_1))) + \ldots$$

Extend the weight degree function d to $\mathbf{C}^{[N+1]}$ by putting $d(\xi) = -d_x$. Then the above form of elements of J and Convention 5.1 imply that $\widehat{J}_d$ is generated by $x\xi - 1, xv_1 - q_0(y, z), v_2 - \xi v^{n_1}$, etc., where $\widehat{J}_d$ is the ideal generated by the principal components of elements of J. Hence $\widehat{J}_d = K[\xi] = K[1/x]$. This means that $\widehat{J}_d$ defines a variety $\widehat{Z} \subset \mathbf{C}^{N+1}$ such that the image of its natural projection to $\mathbf{C}^N$ is $Y \setminus \{x = 0\}$. Since $I' \subset J$ we see that $\widehat{I}'_d \subset \widehat{J}_d$ whence every element of $\widehat{I}'_d$ vanishes on $\widehat{Z}$ and, therefore, on Y. □

Remark 5.3. The variety $\widehat{X}'_d$ is independent on the choice of d satisfying Convention 5.1 and it is isomorphic to the zero set of the following polynomial equations in the space $\mathbf{C}^{3+m}$ with coordinates $(x, y, z, v_1, \ldots, v_m)$

$$P_1(x, y, z, v_1) = xv_1 - q_0(y, z) = 0$$
$$P_2(x, v_1, v_2) = xv_2 - v_1^{n_1} = 0$$
$$\ldots$$
$$P_m(x, v_{m-1}, v_m) = xv_m - v_{m-1}^{n_{m-1}} = 0.$$

Therefore, we shall write further $\widehat{I}', \widehat{A}'$, and $\widehat{X}'$ instead of $\widehat{I}'_d, \widehat{A}'_d$, and $\widehat{X}'_d$ provided it does not cause misunderstanding.

5.3. **Locally nilpotent derivation of Jacobian type.** We shall study locally nilpotent derivations on the associate graded algebra $\widehat{A}' = \mathbf{C}[\widehat{X}']$ which was introduced in the previous subsection. We shall use the presentation of $\widehat{X}'$ given in Remark 5.3. In particular, $P_1, \ldots, P_m$ have the same meaning as in that Remark.

For polynomial $q_1, \ldots, q_{m+3}$ on $\mathbf{C}^{m+3}$ we denote by $J(q_1, \ldots, q_{m+3})$ their Jacobian with respect to $(x, y, z, v_1, \ldots, v_m)$. In this subsection and the next one we denote $q|_{\widehat{X}'}$ by $\tilde{q}$ for every polynomial q on $\mathbf{C}^{m+3}$. Note that $(\tilde{x}, \tilde{y}, \tilde{z})$ is a local (holomorphic) coordinate system at each point of $\widehat{X}_0 = \widehat{X}' \setminus \{x = 0\}$. For $a_1, a_2, a_3 \in \widehat{A}'$ we denote by $J_0(a_1, a_2, a_3)$ the Jacobian of these regular functions on $\widehat{X}_0$ with respect to $\tilde{x}, \tilde{y}$, and $\tilde{z}$. This is a rational function on $\widehat{X}'$ but $\tilde{x}^m J_0(a_1, a_2, a_3)$ is already regular on $\widehat{X}'$ since x^m is the determinant of the matrix $\{\partial P_i/\partial v_j \,|\, i, j = 1, \ldots, m\}$. Furthermore, if $a_i = \tilde{q}_i$ then $J(P_1, \ldots, P_m, q_1, q_2, q_3)|_{\widehat{X}'} = \tilde{x}^m J_0(a_1, a_2, a_3)$ [KaM-L2]

(it is useful to keep this equality in mind when we shall make direct computations in Proposition 5.3).

Definition 5.8. Fix $a_1, a_2 \in \widehat{A}'$ and let $a \in \widehat{A}'$. Then $\partial(a) = \tilde{x}^m J_0(a_1, a_2, a)$ is called a derivation of Jacobian type on $\widehat{A}'$.

Lemma 5.3. [KaM-L2] *Let δ be a nontrivial locally nilpotent derivation on $\widehat{A}'$ and let $a_1, a_2 \in \mathrm{Ker}\,\delta$ be algebraically independent. Then $\partial(a) = \tilde{x}^m J_0(a_1, a_2, a)$ is a locally nilpotent derivation which is equivalent to δ.*

We say that $a \in \widehat{A}'$ is d-homogeneous if a is the restriction to $\widehat{X}'$ of a d-homogeneous polynomial. We are going to study locally nilpotent derivations of Jacobian type on $\widehat{A}'$ such that a_1 and a_2 in the above definition are irreducible d-homogeneous where d is a weight degree function satisfying Convention 5.1.

Lemma 5.4. *Let $a \in \widehat{A}'$ be an irreducible d-homogeneous element. Then up to a constant factor a is of one of the following elements $\tilde{v}_i, \tilde{x}, \tilde{y}, \tilde{z}$, or $\tilde{y}^k + c\tilde{z}^l$ where $c \in \mathbf{C}^*$ and k, l are the same as in Proposition 4.1.*

Proof. Let q be d-homogeneous and $a = \tilde{q}$ (in particular, q is irreducible). It follows from the explicit form of the polynomial system in Remark 5.3 and Convention 5.1 that we can suppose that each monomial from $M(q)$ is non-divisible by xv_i for every $i = 1, \ldots, m$. The restriction of every function v_i to $\widehat{X}'$ is of form q_0^s/x^j where $s, j > 0$. Note that if we extend d naturally to the field of rational functions then $d(v_i) = d(q_0^s/x^j)$ in virtue of Convention 5.1.

Assume that $\mu_1, \mu_2 \in M(q)$ are such that μ_1 is divisible by x but μ_1 is not. Then μ_1 and μ_2 coincides with the restriction to $\widehat{X}'$ of the functions $x^{j_1} y^{\alpha_1} z^{\beta_1}$ and and $y^{\alpha_2} z^{\beta_2} q_0^s/x^{j_2}$ where $j_1 > 0, j_2 \geq 0$. Since $d(\mu_1) = d(\mu_2)$ we have $d(x^{j_1} y^{\alpha_1} z^{\beta_1}) = d(y^{\alpha_2} z^{\beta_2} q_0^s/x^{j_2})$ whence $(j_1 + j_2)d_x = d(y^{\alpha_2 - \alpha_1} z^{\beta_2 - \beta_1} q_0^s)$. Since $d_y = (l/k)d_z$ and $d(q_0) = kd_y$ we see that d_x and d_y are $\mathbf{Q}$-dependent which contradicts Convention 5.1.

Thus the assumption is wrong and if one monomial from $M(q)$ is divisible by x then every monomial is. Therefore, we suppose that none of the monomials from $M(q)$ is divisible by x since we are interested in the case when a is irreducible. Let $\mu_1, \mu_2 \in M(q)$ and $\mu_i = y^{\alpha_i} z^{\beta_i} v_i$ where ν_i is a monomial which depends on $v_1, \ldots, v_m$ only. The restriction of μ_i to $\widehat{X}'$ coincides with $y^{\alpha_i} z^{\beta_i} q_0^{s_i}/x^{j_i}$. Thus

$$d(y^{\alpha_1} z^{\beta_1} q_0^{s_1}/x^{j_1}) = d(y^{\alpha_2} z^{\beta_2} q_0^{s_2}/x^{j_2}).$$

54

The same argument as above shows that $j_1 = j_2$ since otherwise d_x and d_y are $\mathbf{Q}$-dependent. Hence

$$d(y^{\alpha_1} z^{\beta_1} q_0^{s_1}) = d(y^{\alpha_2} z^{\beta_2} q_0^{s_2}).$$

Since $d(q_0) = k d_y = l d_z$ and $(k, l) = 1$ we see that $\alpha_i = \alpha_0 + t_i k$ and $\beta_i = \beta_0 + \tau_i l$ where α_0 is one of the numbers $0, 1, \ldots, k-1$, β_0 is one of the numbers $0, 1, \ldots, l-1$ and $t_1 - t_2 + \tau_1 - \tau_2 = s_2 - s_1$.

Therefore, the restriction of q to $\widehat{X}'$ coincides with

$$y^{\alpha_0} z^{\beta_0} \varphi(y^k, z^l) q_0^s / x^j$$

where $\varphi(y^k, z^l)$ is d-homogeneous and the restriction of q_0^s / x^j to $\widehat{X}'$ coincides with the restriction of some monomial ν which depends on $v_1, \ldots, v_m$ only. Now the statement of Lemma follows from the fact that that $\varphi(y^k, z^l)$ is the product of factors of type $c_1 y^k + c_2 z^l$ where $c_1, c_2 \in C$. $\qquad\square$

Corollary 5.2. *Let* $a = \tilde{q}$ *where* $q \notin \mathbf{C}[y, z]$ *is a* d*-homogeneous polynomial which does not depend on* x*. Then* q *is divisible by some* v_i*.*

Proposition 5.3. *Let* X' *be as in Proposition 4.1. Suppose also that* X' *is smooth. Let* $\partial(a) = \tilde{x}^m J_0(a_1, a_2, a)$ *be a nontrivial locally nilpotent derivation of Jacobian type on* $\widehat{A}'$ *and let* a_1 *and* a_2 *be irreducible* d*-homogeneous. Suppose that* $m \geq 2$*. Then*

(1) (a_1, a_2) *coincides (up to the order) with one of the pairs* $(\tilde{x}, \tilde{y})$ *or* $(\tilde{x}, \tilde{z})$*,*

(2) $\tilde{x} \in \mathrm{Ker}\, \partial$ *and* $\deg_{\partial}(v_i) \geq 2$ *for every* $i = 1, \ldots, m$*.*

Proof. If (a_1, a_2) is one of the pairs in (1) it is easy to check that ∂ is nontrivial and locally nilpotent, and (2) holds also. We need to show that if we use other possible irreducible d-homogeneous elements as a_1, a_2 (recall that such elements were described in Lemma 5.4) then ∂ is not a nontrivial locally nilpotent derivation. Since we want a nontrivial derivation we need to consider only a_1 and a_2 which are algebraically independent in $\widehat{A}'$.

Case 1. Let $(a_1, a_2) = (\tilde{y}, \tilde{z})$. The direct computation shows that $\partial(\tilde{x}) = \tilde{x}^m$ whence ∂ cannot be locally nilpotent. Indeed, one can see that $\deg_{\partial}(\partial(\tilde{x})) = \deg_{\partial}(\tilde{x}) - 1$. On the other hand $\deg_{\partial}(\tilde{x}^m) = m \deg_{\partial}(\tilde{x})$ which yields a contradiction.

Case 2. Either a_1 or a_2 is of form $\tilde{y}^k + c\tilde{z}^l$ where $c \in \mathbf{C}^*$ and k and l are as in Proposition 4.1. By Theorem 5.1 $\tilde{y}, \tilde{z} \in \mathrm{Ker}\, \partial$ since $\tilde{y}^k + c\tilde{z}^l \in \mathrm{Ker}\, \partial$. By Lemma 5.3 ∂ is equivalent to the derivation $\tilde{x}^m J_0(\tilde{y}, \tilde{z}, a)$ whence this case does not hold.

Case 3. Let $(a_1, a_2) = (\tilde{v}_{i_1}, \tilde{v}_{i_2})$ where $i_1 < i_2$. Consider the identical morphism $F : \widehat{X}' \to \widehat{X}' \subset \mathbf{C}^{m+3}$ and morphism $G : \widehat{X}' \to \mathbf{C}^2$ given by $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{v}_1, \ldots, \tilde{v}_m) \to$

$(\tilde{v}_{i_1}, \tilde{v}_{i_2})$. Recall that $\tilde{v}_{i_k} = \tilde{q}_0^{s_k}/\tilde{x}^{j_k}$. It is easy to check that $\tilde{v}_{i_1}$ and $\tilde{v}_{i_2}$ are algebraically independent in $\widehat{A}'$ which means that the pairs (s_1, j_1) and (s_2, j_2) are not proportional (in fact, it can be shown by induction that $j_2 s_1 - j_1 s_2 > 0$). Consider a generic point $\xi \in \mathbf{C}^2$ and the fiber $G^{-1}(\xi)$. Since (s_1, j_1) and (s_2, j_2) are not proportional one can see that each component of this fiber is a curve in $\mathbf{C}^{m+3}$ given by equations $v_i = c_i, x = c'$, and $q_0(y, z) = y^k - z^l = c$ where $c_i, c' \in \mathbf{C}$ and $c \in \mathbf{C}^*$. This curve is hyperbolic and thus it does not admit non-constant morphisms from $\mathbf{C}$. By Theorem 5.1 if ∂ is locally nilpotent then $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{v}_i \in \operatorname{Ker} \partial$ whence ∂ is trivial. Therefore, this case does not hold.

Case 4. Let $(a_1, a_2) = (\tilde{x}, \tilde{v}_i)$. The same argument is in Case 3 works.

Case 5. Let $a_1, a_2) = (\tilde{y}, \tilde{v}_i)$. Consider again the identical morphism $F : \widehat{X}' \to \widehat{X}'$ and morphism $G : \widehat{X}' \to \mathbf{C}^2$ given by $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{v}_1, \ldots, \tilde{v}_m) \to (\tilde{y}, \tilde{v}_i)$. Recall that $\tilde{v}_i$ is of form $\tilde{v}_i = \tilde{q}_0^s/\tilde{x}^j$ where $j \geq 2$ if $i > 1$. This implies that the curve $G^{-1}(\xi)$ where $\xi = (c_1, c_2) \in \mathbf{C}^2$ is isomorphic to the curve $(c_1^k - z^l)^s - c_2 x^j = 0$. When $j \geq 2$ and s is not divisible by j the last curve does not have contractible components for generic ξ. Theorem 5.1 (6) implies that if in this case ∂ is locally nilpotent then ∂ must be trivial. If $j \geq 2$ and s is divisible by j then each irreducible component of $G^{-1}(\xi)$ is contractible and contains double points of $G^{-1}(\xi)$. Since $G^{-1}(\xi) \subset \widehat{X}'$ is invariant under the $\mathbf{C}_+$-action generated by the locally nilpotent derivation ∂ the singular points of this curve must be fixed under this action whence the action itself is trivial on $G^{-1}(\xi)$. Therefore, it is trivial on $\widehat{X}'$ whence ∂ is again trivial.

It remains to consider the case when $j = 1$, i.e. $(a_1, a_2) = (\tilde{y}, \tilde{v}_1) = (\tilde{y}, \tilde{q}_0/\tilde{x})$. The direct computation shows that $\partial(\tilde{x})$ coincides (up to a constant multiple) with $\tilde{x}^{m-1}\tilde{z}^{l-1}$. Since $m \geq 2$ we see that ∂ cannot be nontrivial locally nilpotent (indeed, compare $\deg_{\partial}(\tilde{x})$ and $\deg_{\partial}(\partial(\tilde{x}))$) and we have to disregard this case.

Case 6. When $(a_1, a_2) = (\tilde{z}, \tilde{v}_i)$ the same argument as in Case 5 works. $\square$

5.4. The computation of $\operatorname{ML}(A')$.

By Definition 5.4 each nontrivial $\partial \in \operatorname{LND}(A')$ generates a nontrivial $\widehat{\partial}_d \in \operatorname{LND}(\widehat{A}')$ which depends on the choice of the weight degree function d. Similarly the mapping $\operatorname{gr}_d : A' \to \widehat{A}'$ depends on d. The following relation between ∂ and $\widehat{\partial}_d$ is simple but essential for us

$$\text{if } \deg_{\partial} = s \text{ then } \deg_{\widehat{\partial}_d}(\operatorname{gr}_d(a))(a) \leq s.$$

Definition 5.9. A locally nilpotent derivation ∂ on A' is called perfect if its associate derivation $\widehat{\partial}_d$ is of form $\widehat{\partial}_d(a) = \tilde{x}^m J_0(a_1, a_2, a)$ where $a_1, a_2 \in \widehat{A}'_d$ are irreducible

d-homogeneous and algebraically independent. The set of all perfect derivations will be denoted by $\partial \in \mathrm{Per}\,(A')$.

Proposition 5.4. [KaM-L2] *If the associate graded algebra $\widehat{A}'_d$ is a domain (i.e., $\widehat{I}'_d$ is prime) then for every nontrivial locally nilpotent derivation on $\widehat{A}'$ there exists an equivalent perfect derivation. In particular,*

$$\mathrm{ML}(A') = \bigcap_{\partial \in \mathrm{Per}\,(A')} \mathrm{Ker}\,\partial.$$

Proposition 5.5. *Let A' be as in Proposition 4.1 and let d satisfy Convention 5.1. Suppose that X' is smooth. For every $\partial \in \mathrm{LND}(A')$ we have $x \in \mathrm{Ker}\,\partial$ whence $\mathrm{ML}(A') \neq \mathbf{C}$.*

Proof. Since A' is a domain by Proposition 5.2, it is enough to consider $\partial \in \mathrm{Per}\,(A')$ by Proposition 5.4. Consider $a \in A'$ with $\deg_\partial(a) \leq 1$. Show that there exist a polynomial q with $q|_{X'} = a$ such that none of monomial $\mu \in M(q)$ is divisible by v_i or $u_{s,j}$ for all i, s, j (i.e. $q \in \mathbf{C}[x, y, z]$). It follows from the explicit from of the polynomial system in Proposition 4.1 that we can suppose that none of $\mu \in M(q)$ is divisible by xv_i or $xu_{s,j}$. Thus $M(q)$ is the disjoint union $M_1(q) \cup M_2(q)$ where $M_1(q)$ consists of monomials which depends on x, y, z only and $M_2(q)$ consists of monomials which do not depend on x and do not belong to $\mathbf{C}[y, z]$. Show that $M_2(q)$ is empty. Let $\mu \in M_2(q)$. Under Convention 5.1 one can keep d_y, d_z fixed, decrease d_x, and increase $d_i, d_{s,j}$ so that $d(\mu) > d(\nu)$ for every $\nu \in M_1(q)$. Hence if $\bar{q}_d$ is the principal component of q then $M(\bar{q}_d) \subset M_2(q)$. The relation between ∂ and $\widehat{\partial}_d$ shows that $\deg_{\widehat{\partial}_d}(\mathrm{gr}_d(a)) \leq 1$. The element $\mathrm{gr}_d(a) = \bar{q}_d|_{\widehat{X}'}$ is the product of irreducible d-homogeneous elements of $\widehat{A}'$. By Corollary 5.2 one of these elements is $\tilde{v}_i$ whence $\deg_{\widehat{\partial}_d}(\tilde{v}_i) \leq \deg_{\widehat{\partial}_d}(\mathrm{gr}_d(a)) \leq 1$ which contradicts Proposition 5.3. Thus $M_2(q)$ is empty. Let $b \in A'$ with $\deg_\partial(b) = 1$. Recall that by Theorem 5.1 there exist $a', a_0, \ldots, a_s \in \mathrm{Ker}\,\partial$ such that $a'\tilde{v}_1 = \sum_{j=0}^s a_j b^j$ where $s = \deg_\partial \tilde{v}_i$. Hence $\tilde{v}_1 = (q(x, y, z)/r(x, y, z))|_{X'}$ where $a' = r(x, y, z)|_{X'}$. On the other hand $A' \subset \mathbf{C}[x, y, z, 1/f(x)]$ where f is as in Proposition 4.1. Since $v_1 \notin \mathbf{C}[x, y, z]$ we see that $r(x, y, z)$ must be divisible by some $x - c$ where c is a root of f. Therefore, $x - c \in \mathrm{Ker}\,\partial$ as a divisor of an element from $\mathrm{Ker}\,\partial$ whence $x \in \mathrm{Ker}\,\partial$. $\qquad\square$

This implies

Lemma II. *Let X' be an affine algebraic variety of dimension 3 such that X' is a UFD, all invertible functions on X' are constants, and let the assumption $(1')$ and*

(2′) of Lemma I hold, but (3) does not. Then X' is an exotic algebraic structure on $\mathbf{C}^3$ with a nontrivial Makar-Limanov invariant.

Remark 5.4. In [KaZa1] we described some conditions under which one can extend locally nilpotent derivations from A to A'. Using this technique it is not difficult to show that $\mathrm{ML}[A'] = \mathbf{C}[x]|_{X'}$.

Lemmas I and II imply

Theorem 5.2. Let X' be an affine algebraic variety of dimension 3 such that X' is a UFD , all invertible functions on X' are constants, and

(1) the Euler characteristic of X' is $e(X') = 1$;

(2′) there exists a Zariski open subset Z of X' which is a $\mathbf{C}^2$ -cylinder over a curve U ;

(3) each irreducible component of $X' \setminus Z$ is a UFD.

Then U is isomorphic to a Zariski open subset of $\mathbf{C}$ and p can be extended to a regular function on X' . Furthermore, X' is isomorphic to $\mathbf{C}^3$ and p is a variable.

The same conclusion remains true if we replace (1) and (3) by

(1′) X' is smooth and $H_3(X') = 0$;

(3′) each irreducible component of $X' \setminus Z$ has at most isolated singularities.

In the case when conditions (1′) and (2′) hold but (3) does not, X' is an exotic algebraic structure on $\mathbf{C}^3$ (that is, X' is diffeomorphic to $\mathbf{R}^6$ as a real manifold but not isomorphic to $\mathbf{C}^3$) with a nontrivial Makar-Limanov invariant.

In order to finish the proof of the Main Theorem we need now to show that condition (2′) above is equivalent to condition (2) in the Main Theorem. This will be done in [KaZa2].

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